Appendix A

Injective Modules and Matlis Duality

Notes on "24 Hours of Local Cohomology"
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We take R to be a commutative ring, and will discuss the theory of injective R-modules. The following facts are commonly used results:

(1) For any R-module M, there is a unique smallest injective module containing M, denoted $E_R(M)$ and called the *injective hull*. It has the property that any injective module containing M has $E_R(M)$ as a submodule.

(2) Any injective module over a noetherian ring has a unique direct sum decomposition into indecomposable injective modules.

(3) The indecomposable injective modules of a noetherian ring are of the form $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$.

(4) **Matlis Duality** Suppose $(R, \mathfrak{m})$ is a complete local ring, $E = E_R(R/\mathfrak{m})$, and $(-)^\vee$ is the R-module functor $\operatorname{Hom}_R(-, E)$. Then

  (a) If M is noetherian, then $M^\vee$ is artinian.

  (b) If M is artinian, then $M^\vee$ is noetherian.

  (c) If M is noetherian or artinian, then $M \cong M^{\vee\vee}$.

Note: We say a module M is *noetherian* if it satisfies the ascending chain condition on submodules, and that it is *artinian* if it satisfies the descending chain condition on submodules.

A.1 Essential Extensions

Definition A.1. An R-module E is *injective* if the functor $\operatorname{Hom}_R(-, E)$ is exact; i.e. if for every injective map of R-modules $f : M \to N$ and R-module homomorphism $g : M \to E$, there exists $h : N \to E$ such that $g = h \circ f$.

Theorem A.2 (Baer's criterion)**.** *An R-module E is injective if and only if for every ideal $\mathfrak{a} \subseteq R$ and R-module homomorphism $f : \mathfrak{a} \to E$, there exists an R-module homomorphism $g : R \to E$ such that $g|_{\mathfrak{a}} = f$.*

Note that since the zero homomorphism can always be extended (trivially), we may consider only the nonzero ideals $\mathfrak{a}$ and nonzero homomorphsims $\mathfrak{a} \to E$ when confirming the injectivity of an R-module using Baer's Criterion.

Exercise A.3. *If E and E' are R-modules, then $E \oplus E'$ is injective if and only if E and E' are injective.*

Solution. Suppose $E \oplus E'$ is injective. Let $\mathfrak{a} \subseteq R$ be an ideal and $f : \mathfrak{a} \to E$ be an R-module homomorphism. Then $f \oplus 0 : \mathfrak{a} \to E \oplus E'$ is an R-module homomorphism, and therefore there exists $g : R \to E \oplus E'$ extending $f \oplus 0$. Therefore $\pi_1 \circ g : R \to E$ is a homomorphism, and for $x \in \mathfrak{a}$, $(\pi_1 \circ g)(x) = \pi_1(g(x)) = \pi_1(f(x), 0) = f(x)$, so $\pi_1 \circ g$ extends f. Therefore E is injective by Baer's criterion. By a symmetric argument, so is E'.

Suppose E and E' are injective, $\mathfrak{a} \subseteq R$ is an ideal, and $f : \mathfrak{a} \to E \oplus E'$ is an R-module homomorphism. Then $\pi_1 \circ f : \mathfrak{a} \to E$ and $\pi_2 \circ f : \mathfrak{a} \to E'$ are R-module homomorphisms, hence they extend to $g : R \to E$ and $g' : R \to E'$. Therefore $g \oplus g' : R \to E \oplus E'$ is an R-module homomorphism, and for $x \in \mathfrak{a}$, $(g \oplus g')(x) = (g(x), g'(x)) = (\pi_1(f(x)), \pi_2(f(x))) = f(x)$. Hence $g \oplus g'$ extends f, and so $E \oplus E'$ is injective by Baer's Criterion. ☐

If R is a domain, we say that an R-module M is *divisible* if for all $r \in R \setminus \{0\}$, $r \cdot M = M$, i.e. for every $m \in M$, there exists $m' \in M$ such that $m = r \cdot m'$.

Exercise A.4 (Part 1)**.** *If R is a domain, then all injective R-modules are divisible, and if in addition R is a P.I.D., then all divisible R-modules are injective.*

Solution. Suppose R is a domain and E is an injective R-module. For any $r \in R \setminus \{0\}$ and $e \in E$ we can define a map $f : rR \to E$ by setting $f(r) = e$ and extending linearly. Since E is injective there exists an R-module homomorphism $g : R \to E$ extending f, and $r \cdot g(1) = g(r) = e$. Therefore E is divisible.

Suppose that R is a P.I.D. and M is a divisible module. Let $\mathfrak{a} = aR \subseteq R$ be a nonzero ideal of R and let $f : aR \to M$ be an R-module homomorphism. Since M is divisible we can find $m \in M$ such that $a \cdot m = f(a)$. Let $g : R \to M$ be defined by $g(r) = r \cdot m$. Then g is an R-module homomorphism and we have that $g(a) = a \cdot m = f(a)$. Therefore g extends f, and so M is injective. ☐

As an immediate application of the above exercise, we have that $\mathbb{Q}/d\mathbb{Z}$ is an injective $\mathbb{Z}$-module for $d \in \mathbb{Z}$.

Exercise A.4 (Part 2)**.** *Every nonzero abelian group has a nonzero homomorphism to* $\mathbb{Q}/\mathbb{Z}$*, and if we let* $(-)^\vee = \mathrm{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z})$*, then for each* $\mathbb{Z}$*-module* M*, the natural map* $M \to M^{\vee\vee}$ *is injective.*

Solution. Let G be a nonzero abelian group, i.e. a nonzero $\mathbb{Z}$-module. Let $x \in G$ and let $H = \mathbb{Z} \cdot x \subseteq G$ be a submodule of G. Define $f : H \to \mathbb{Q}/\mathbb{Z}$ as follows: if $n \cdot x \neq 0$ for $n > 0$, then $f(x) = \frac{1}{2} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Otherwise, let n be the smallest positive integer such that $n \cdot x = 0$, and set $f(x) = \frac{1}{n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Then f is a well-defined $\mathbb{Z}$-module homomorphism from H to $\mathbb{Q}/\mathbb{Z}$. But since we have an injective map (namely inclusion) from H to G and $\mathbb{Q}/\mathbb{Z}$ is injective (by the above exercise), there exists an R-module homomorphism $g : G \to \mathbb{Q}/\mathbb{Z}$ extending f. Therefore g is nonzero, and the first statement is proved. In particular we have proved more: given any nonzero $x \in G$ we can find a $\mathbb{Z}$-module homomorphism $g : G \to \mathbb{Q}/\mathbb{Z}$ such that $g(z) \neq 0$.

Now let M be any $\mathbb{Z}$-module. The natural map

$$M \to M^{\vee\vee} = \mathrm{Hom}_\mathbb{Z}(\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

takes $m \mapsto (f \mapsto f(m))$. Suppose $0 \neq m \in M$. Then by the last sentence of the previous paragraph, we can find $f \in \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ such that $f(m) \neq 0$. Therefore m is not in the kernel of the map $M \to M^{\vee\vee}$. Therefore the kernel of this map is 0, i.e. the map is injective. □

Exercise A.5. *Let* R *be an* A*-algebra. Then*

(1) *If* E *is an injective* A*-module and* F *is a flat* R*-module, then* $\mathrm{Hom}_A(F, E)$ *is an injective* R*-module; and*

(2) *Every* R*-module embeds in an injective* R*-module.*

The solution is below is modified from Mel Hochster's local cohomology notes, page 6.

Solution. By definition, $\mathrm{Hom}_A(F, E)$ is an injective R-module if the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_A(F, E))$ is exact. For any R-module M, $\mathrm{Hom}_R(M, \mathrm{Hom}_A(F, E)) \cong \mathrm{Hom}_A(M \otimes_R F, E)$ by the adjointness of Hom and $\otimes$. Since F is a flat R-module, the functor $- \otimes_R F$ is exact, and since E is an injective A-module, the functor $\mathrm{Hom}_A(-, E)$ is exact. Therefore $\mathrm{Hom}_R(-, \mathrm{Hom}_A(F, E)) \cong \mathrm{Hom}_A(- \otimes_R F, E)$ is the composition of two exact functors, hence is exact. This proves part (1).

Let M be an R-module. Letting $(-)^\vee$ be the functor $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z})$, we have that $M \to M^{\vee\vee}$ is injective by exercise A.4(4). Let F be a flat (e.g. free) R-module and f an R-module homomorphism such that $f : F \to \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = M^\vee$ is surjective. Since $(-)^\vee$ is a left-exact contravariant functor on R-modules, $f^\vee : M^{\vee\vee} \to F^\vee$ is injective. Therefore M embeds in $F^\vee$. But since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and R is a $\mathbb{Z}$-algebra (since any ring is), $F^\vee = \mathrm{Hom}_\mathbb{Z}(F, \mathbb{Q}/\mathbb{Z})$ is an injective R-module by part (1). □

Proposition A.6. *Let* $\theta : M \to N$ *be an injective map of* R*-modules. The following are equivalent:*

(1) For any homomorphism $\epsilon : N \to Q$, if $\epsilon \circ \theta$ is injective, then so is ϵ.

(2) Every nonzero submodule of N has nonzero intersection with $\theta(M)$.

(3) Every nonzero element of N has a nonzero multiple in $\theta(M)$.

Proof. Since θ is injective, we can identify M with $\theta(M)$ and write $M \subseteq N$. In this context, $\epsilon \circ \theta = \epsilon|_M$.

(1) $\Rightarrow$ (2): Let N' be a nonzero submodule of N. Then the quotient map $\epsilon : N \to N/N'$ is not injective, and so we must have that $\epsilon|_M$ is not injective. Therefore there exists $0 \neq m \in M$ such that $\epsilon(m) = 0 + N'$, i.e. $m \in N'$.

(2) $\Rightarrow$ (3): Let $0 \neq n \in N$ and consider the submodule $R \cdot n \subseteq N$. By our hypothesis, there exists $0 \neq m \in M \cap R \cdot n$, i.e. n has a nonzero multiple in M.

(3) $\Rightarrow$ (1): Let $\epsilon : N \to Q$ be an R-module homomorphism such that $\epsilon|_M$ is injective. Let $0 \neq n \in N$, and take $r \in R$ such that $0 \neq r \cdot n \in M$. Then $\epsilon(r \cdot n) = \epsilon|_M(r \cdot n) \neq 0$. Therefore $r \cdot n \neq 0$, i.e. $n \neq 0$. Therefore ϵ is injective.

For good measure we prove some additional implications

(2) $\Rightarrow$ (1) Let $\epsilon : N \to Q$ be an R-module homomorphism such that $\epsilon|_M$ is injective. If $\ker \epsilon \neq 0$, then there exists $0 \neq m \in \ker \epsilon \cap M = \ker \epsilon|_M = 0$, a contradiction. Therefore ϵ is injective.

(3) $\Rightarrow$ (2) Suppose $N' \subseteq N$ is a nonzero submodule. Then there exists $0 \neq n \in N'$, and so there exists $r \in R$ such that $r \cdot n \in M$. Since $r \cdot n \in N'$ also, we have that $M \cap N' \neq 0$. □

Definition A.7. If $\theta : M \hookrightarrow N$ satisfies any of the conditions of the above proposition, we say that N is an *essential extension* of M.

Example A.8. If $U \subseteq R$ is a multiplicative system of nonzerodivisors of R then $U^{-1}R$ is an essential extension of R.

That U contain only zerodivisors of R is necessary for the natural map $R \to U^{-1}R$ to be injective. If $0 \neq \frac{r}{u} \in U^{-1}R$ then $u \cdot \frac{r}{u} = \frac{r}{1}$ and $\frac{r}{1}$ is in the image of R, so $U^{-1}R$ is an essential extension of R.

Example A.9. Let $(R, \mathfrak{m})$ be a local ring and N an $\mathfrak{m}$-torsion R-module (this means that every element of N is killed by some power of $\mathfrak{m}$). Then the *socle* of N is denoted and defined as $\operatorname{soc}(N) = (0 :_N \mathfrak{m})$. The extension $\operatorname{soc}(N) \subseteq N$ is essential: If $0 \neq y \in N$, let t be the smallest integer such that $\mathfrak{m}^t y = 0$. Then $\mathfrak{m}^{t-1} y = 0$, i.e. y has a nonzero multiple in $\operatorname{soc}(N)$.

Exercise A.10. *Let I be an indexing set, and for each $i \in I$ let M_i and N_i be R-modules. Then $\bigoplus_{i \in I} M_i \subseteq \bigoplus_{i \in I} N_i$ is essential if and only if $M_i \subseteq N_i$ is essential for each $i \in I$.*

Solution. Suppose that $\bigoplus_{i \in I} M_i \subseteq \bigoplus_{i \in I} N_i$ is essential, and fix $j \in I$. Let $0 \neq L_j \subseteq N_j$ be an R-module. Then $L = L_j \oplus \bigoplus_{i \in I \setminus \{j\}} 0$ is a nonzero submodule of $\bigoplus_{i \in I} N_i$. Therefore

$$0 \neq L \cap \bigoplus_{i \in I} M_i = (L_j \cap M_j) \oplus \bigoplus_{i \in I \setminus \{j\}} 0,$$

and so $L_j \cap M_j \neq 0$. Hence $M_j \subseteq N_j$ is essential.

Now suppose that $M_i \subseteq N_i$ is essential for each $i \in I$. Let $0 \neq L \subseteq \bigoplus_{i\in I} N_i$. Then $L = \bigoplus_{i\in I} L_i$, where $L_i = L \cap N_i$. Choose $i \in I$ such that $L_i \neq 0$. Then since $M_i \subseteq N_i$ is essential, $M_i \cap L_i \neq 0$, hence $\bigoplus_{i\in I} M_i \cap L \neq 0$. Therefore $\bigoplus_{i\in I} M_i \subseteq \bigoplus_{i\in I} N_i$ is essential. □

Exercise A.11. *Let k be a field, $R = k[[x]]$, and $N = R_x/R$. Then $\operatorname{soc}(N) \subseteq N$ is essential and $\prod_{\mathbb{N}} \operatorname{soc}(N) \subseteq \prod_{\mathbb{N}} N$ is not essential.*

Solution. The ring R is local with maximal ideal $\mathfrak{m} = (x)$ and the elements of N are of the form $\frac{f}{x^k} + R$ for some $k \in \mathbb{N}$ and $f \in k[x]$ with degree less than k. For any such element, $\mathfrak{m}^k = (x^k)$ kills that element. Therefore N is $\mathfrak{m}$-torsion, an so by example $A.9$, $\operatorname{soc}(N) \subseteq N$ is essential.

Now consider the extension $\prod_{\mathbb{N}} \operatorname{soc}(N) \subseteq \prod_{\mathbb{N}} N$. Let $y \in \prod_{\mathbb{N}} N$ be given by $y_i = \frac{1}{x^i} + R$. Let $0 \neq \sum_{i\in\mathbb{N}} g_i \in R$ be any nonzero element of R. Let $j \in \mathbb{N}$ such that $g \in (x^j)\backslash(x^{j+1})$. Then the $j+2$th coordinate of $g \cdot y$ is $\frac{g}{x^{j+2}} + R = \frac{g_j + g_{j+1}x}{x^2} + R$. But this element is not in $\operatorname{soc}(N)$ since $x \cdot \left(\frac{g_j + g_{j+1}x}{x^2} + R\right) = \frac{g_j}{x} + R \neq 0$. Therefore $g \cdot y \notin \prod_{\mathbb{N}} \operatorname{soc}(N)$, and so $\prod_{\mathbb{N}} \operatorname{soc}(N) \subseteq \prod_{\mathbb{N}} N$ is not essential. □

Proposition A.12. *Let $L \subseteq M \subseteq N$ be nonzero R-modules.*

(1) $L \subseteq N$ is essential if and only if $L \subseteq M$ and $M \subseteq N$ are essential.

(2) Let I be an index set and let N_i be R-modules such that for each i, $M \subseteq N_i \subseteq N$ and $\bigcup_i N_i = N$. Then $M \subseteq N$ is essential if and only if each $M \subseteq N_i$ is essential.

(3) There exists a unique module $N' \subseteq N$ with $M \subseteq N' \subseteq N$ maximal with respect to the property that $M \subseteq N'$ is essential.

Proof. (1) Suppose $L \subseteq N$ is essential. Let $0 \neq M' \subseteq M \subseteq N$ and $0 \neq N' \subseteq N$. Then $M' \cap L \neq 0$ since $L \subseteq N$ is essential and $N' \cap M \supseteq N' \cap L \neq 0$ since $L \subseteq N$ is essential. Therefore $L \subseteq M$ and $M \subseteq N$ are essential. Now suppose $L \subseteq M$ and $M \subseteq N$ are essential and let $0 \neq N' \subseteq N$. Then $M \cap N' \subseteq M \neq 0$ since $M \subseteq N$ is essential, and therefore $L \cap (M \cap N') \neq 0$ since $L \subseteq M$ is essential. But since $L \cap N' \supseteq L \cap (M \cap N')$, we're done.

(2) Suppose $M \subseteq N$ is essential. Fix $i \in I$ and let $0 \neq N'_i \subseteq N_i$. Then $N'_i \subseteq N$, and since $M \subseteq N$ is essential, $M \cap N'_i \neq 0$. Therefore $M \subseteq N_i$ is essential. Now suppose $M \subseteq N_i$ is essential for all $i \in I$. Let $0 \neq N' \subseteq N$. Since $N = \bigcup_i N_i$, there exists $i \in I$ such that $N' \cap N_i \neq 0$. Since $M \subseteq N_i$ is essential, $M \cap N' \supseteq M \cap (N' \cap N_i) \neq 0$. Therefore $M \subseteq N$ is essential.

(3) Let $\mathcal{S} = \{S \subseteq N \mid M \subseteq S \text{ is essential}\}$ and order $\mathcal{S}$ by inclusion. Let $S_1 \subseteq S_2 \subseteq \cdots$ be an increasing sequence of elements of $\mathcal{S}$. Then since each S_i is essential over M, so is $S_\infty = \bigcup_{i\in\mathbb{N}} S_i$, and we have that $S_\infty \supset S_i$ for all $i \in \mathbb{N}$. Therefore S_∞ is a maximal element for the increasing sequence, and by Zorn's Lemma, there exist maximal elements of $\mathcal{S}$. Let S, S' be two maximal elements of $\mathcal{S}$. Then since $M \subseteq S$ and $M \subseteq S'$ are both essential extensions, so

is $M \subseteq S \cup S'$. So $S \cup S' \in \mathcal{S}$ and $S \cup S' \supseteq S, S'$, and so by the maximality of S and S' we must have that $S = S \cup S' = S'$. Therefore the maximal element of $\mathcal{S}$ is unique. □

Definition A.13. The module N' in Proposition A.12(3) is called the *maximal essential extension of M in N*. If $M \subseteq N$ is essential and N has no proper essential extensions, then N is called a *maximal essential extension of M*.

Proposition A.14. *Let M be an R-module. The following are equivalent:*

(1) M is injective;

(2) every injective homomorphism $M \hookrightarrow N$ splits;

(3) M has no proper essential extensions

Proof. (1) ⇒ (2): Let $f : M \to N$ be an injective homomorphism. Since M is injective and $\mathrm{id}_M : M \to M$ is a homomorphism, id_M extends to a map $g : N \to M$. Therefore g splits f.

(2) ⇒ (3): Suppose $M \subseteq N$ is an essential extension. Then there is a splitting $f : N \to M$ of the inclusion map ι. Since $f \circ \iota$ is the identity map, it is injective, and so since $M \subseteq N$ is essential, f is injective. Therefore f is bijective and so $M = N$.

(3) ⇒ (1): By Exercise A.5 we can embed M in an injective module E. If $N_1 \subseteq N_2 \subseteq \cdots$ is an increasing sequence of submodules of E such that $N_i \cap M = 0$ for all i, then $M \cap \bigcup_{i \in \mathbb{N}} N_i = 0$. Therefore there exists a submodule N of E maximal with respect to the property of having zero intersection with M by Zorn's Lemma. So now $f : M \to E/N$ is an essential extension, hence $M \cong E/N$. Therefore $E = M + N$ and since $M \cap N = 0$, we have that $E = M \oplus N$. Since M is a direct summand of an injective module, M is itself injective. □

Note in particular that this means that if E is an injective module and $E \subseteq M$ for some R-module M, then $M \cong E \oplus F$ for some R-module F.

Proposition A.15. *Let M be an R-module. If $M \subseteq E$ with E injective, then the maximal essential extension of M in E is an injective module, hence a direct summand of E. Maximal essential extensions of M are isomorphic.*

Definition A.16. The maximal essential extension of M is called the *injective hull* of M and is denoted $E_R(M)$ (or $E(M)$ if the ring is understood).

Definition A.17. Let M be an R-module. An *injective resolution* of M is a complex of injective R-modules

$$0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots$$

with $H^0(E^\bullet) = M$ and $H^i(E^\bullet) = 0$ for $i \geq 1$; it is called *minimal* if $E^0 = E_R(M)$, $E^1 = E_R(E^0/M)$, and E^{i+1} is the injective hull of $\operatorname{coker} \partial^i$ for each $i \geq i$.

A.2 Noetherian rings

We can characterize noetherian rings using injective modules

Proposition A.18. *A ring R is noetherian if and only if every direct sum of injective R-modules is injective.*

We have already seen that the direct sum of two injective modules (and hence of finitely many) is injective; the content of the above proposition lies in the infinite direct sum case. The only part of the proof that we need to fill in the details of is the following: Given a chain of ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ of R and setting $\mathfrak{a} = \bigcup_{i\in\mathbb{N}} \mathfrak{a}_i$, we have natural homomorphisms $\mathfrak{a} \hookrightarrow R \twoheadrightarrow R/\mathfrak{a}_i \hookrightarrow E_R(R/\mathfrak{a}_i)$. From these we have a homomorphism $\mathfrak{a} \to \prod_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)$. We need to show that this map factors through $\bigoplus_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)$. Indeed, let $x \in \mathfrak{a}$. Then there exists $j \in \mathbb{N}$ such that $x \in \mathfrak{a}_i$ for all $i \geq j$. So for any $i \geq j$, the map $R \twoheadrightarrow R/\mathfrak{a}_i$ takes x to 0. Therefore the image of x in $\prod_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)$ is zero for indices greater than or equal to j, i.e. $x \in \bigoplus_{i\in\mathbb{N}} E_R(R/\mathfrak{a}_i)$. This completes the last part of the proof of the proposition.

Definition A.19. An R-module M is called $\mathfrak{a}$-torsion for an ideal $\mathfrak{a}$ of R if every element of M is killed by some power of $\mathfrak{a}$.

Theorem A.20. *Let $\mathfrak{p}$ be a prime ideal of a noetherian ring R and set $\kappa = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, the fraction field of $R_\mathfrak{p}$. Let $E = E_R(R/\mathfrak{p})$.*

(1) If $x \in R \setminus \mathfrak{p}$ then $E \xrightarrow{x} E$ is an isomorphism, hence $E = E_\mathfrak{p}$;

(2) $(0 : E_\mathfrak{p}) = \kappa$;

(3) $\kappa \subseteq E$ is an essential extension of $R_\mathfrak{p}$-modules and $E = E_{R_\mathfrak{p}}(\kappa)$;

(4) E is $\mathfrak{p}$-torsion and $\operatorname{Ass}(E) = \{\mathfrak{p}\}$;

(5) $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E) \cong \kappa$ and $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E_R(R/\mathfrak{q})_\mathfrak{p}) = 0$ for primes $\mathfrak{q} \neq \mathfrak{p}$.

Injective modules, for all their foibles, have a structure theorem when R is noetherian.

Theorem A.21. *Let R be an injective module over a noetherian ring R. There exists a direct sum decomposition*

$$E \cong \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}$$

and the numbers $\mu_\mathfrak{p}$ are independent of the decomposition.

Proposition A.22. *Let $U \subseteq R$ be a multiplicative set.*

(1) If R is an injective R-module, the $U^{-1}R$-module $U^{-1}E$ is injective (injective modules localize).

(2) *If $M \subseteq N$ is a (maximal) essential extension of R-modules, then $U^{-1}M \subseteq U^{-1}N$ is a (maximal) essential extension of $U^{-1}R$-modules (essential extensions localize).*

(3) *The indecomposable injectives over $U^{-1}R$ are the modules $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap U = \emptyset$.*

Definition A.23. Let M be an R-module and $E^\bullet$ its minimal injective resolution. For each i, we have a decomposition

$$E^i = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}.$$

The number $\mu_i(\mathfrak{p}, M)$ is called the *ith Bass number of M with respect to $\mathfrak{p}$*. It is well-defined according to the following theorem:

Theorem A.24. *Let R be a noetherian ring and M an R-module. Let $\mathfrak{p}$ be a prime ideal, and set $\kappa = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then*

$$\mu_i(\mathfrak{p}, M) = \operatorname{rank}_\kappa \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa, M_\mathfrak{p}).$$

We call a homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ a *local* homomorphism if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Theorem A.25. *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a local homomorphism of local rings. If S is module-finite over R, then $\operatorname{Hom}_R(S, E_R(k)) = E_S(\ell)$.*

Remark A.26. Let $(R, \mathfrak{m}, k)$ be a local ring. Then the above theorem states that for any ideal $\mathfrak{a}$ of R we have that $E_{R/\mathfrak{a}}(k) = (0 :_{E_R(k)} \mathfrak{a})$, and since $E_R(k)$ is $\mathfrak{m}$-torsion, we have

$$E_R(k) = \bigcup_{t \in \mathbb{N}} (0 :_{E_R(k)} \mathfrak{m}^t) = \bigcup_{t \in \mathbb{N}} E_{R/\mathfrak{m}^t}(k).$$

A.3 Artinian rings

We define the *length* of a module (denoted $\ell(M)$) as the supremum of the lengths of a composition series $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M$. If $0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \to 0$ is a short exact sequence of R-modules then $\ell(M_2) = \ell(M_1) + \ell(M_2)$. If R is a local ring, then the residue field is the only simple module of R and so the subquotients in a composition series for M are all isomorphic to it.

Lemma A.27. *Let R be a local ring with residue field k, set $(-)^\vee = \operatorname{Hom}_R(-, E_R(k))$, and let M be an R-module. Then*

(1) *The natural map $M \to M^{\vee\vee}$ is injective;*

(2) *$\ell(M^\vee) = \ell(M)$*

In particular, point (2) implies that $M^\vee = 0$ if and only if $M = 0$, i.e. that $(-)^\vee$ is a faithful functor.

Corollary A.28. *$\ell(E_R(k)) = \ell(R)$.*

In particular, $E_R(k)$ has finite length if and only if R is artinian.

Theorem A.29. *Let R be a local ring. Then R is an injective R-module if and only if it is artinian and $\operatorname{rank}_k \operatorname{soc}(R) = 1$.*

A.4 Matlis duality

Remark A.30. Let $\mathfrak{a} \subseteq R$ be an ideal and let M be an R-module. Then there are natural surjections

$$\cdots \longrightarrow M/\mathfrak{a}^3 M \longrightarrow M/\mathfrak{a}^2 M \longrightarrow M/\mathfrak{a}M \longrightarrow 0.$$

The $\mathfrak{a}$-adic completion of M, denoted $\widehat{M}$, is the inverse limit of this system:

$$\varprojlim_i (M/\mathfrak{a}^i M) = \left\{ (..., \overline{m_2}, \overline{m_1}) \in \prod_i M/\mathfrak{a}^i M \mid m_{i+1} - m_i \in \mathfrak{a}^i M \right\}$$

There is a canonical homomorphism of R-modules $M \to \widehat{M}$. The salient properties of this construction are summarized below:

(1) $\ker(M \to \widehat{M}) = \cap_{i\in\mathbb{N}} \mathfrak{a}^i M$;

(2) $\widehat{R}$ is a ring and $R \to \widehat{R}$ is a ring homomorphism;

(3) $\widehat{M}$ is an $\widehat{R}$-module and $M \to \widehat{M}$ is compatible with these structures.

When R is noetherian, then also

(4) The ring $\widehat{R}$ is noetherian;

(5) If $(R, \mathfrak{m})$ is local, then $\widehat{R}$ is local with maximal ideal $\mathfrak{m}\widehat{R}$;

(6) If M is finitely generated, then $\widehat{M} = \widehat{R} \otimes_R M$, and so $\widehat{M}$ is a finitely generated $\widehat{R}$-module;

(7) One has $\widehat{R}/\mathfrak{a}^i\widehat{R} \cong R/\mathfrak{a}^i$ for each i. If M is $\mathfrak{a}$-torsion, then it has a natural $\widehat{R}$-module structure and the map $M \to \widehat{R} \otimes_R M$ is an isomorphism;

(8) If M and N are $\mathfrak{a}$-torsion, then $\operatorname{Hom}_{\widehat{R}}(M, N) = \operatorname{Hom}_R(M, N)$.

Theorem A.31. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\widehat{R}$ ists $\mathfrak{m}$-adic completion, and set $E = E_R(k)$. Then $E_{\widehat{R}}(k) = E$, and the map $\widehat{R} \to \operatorname{Hom}_R(E, E)$ taking r to the homomorphism $e \mapsto r \cdot e$ ("multiplication by r") is an isomorphsim.*

Corollary A.32. *For a local ring $(R, \mathfrak{m}, k)$, the module $E_R(k)$ is artinian.*

Theorem A.33. *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. The following conditions are equivalent:*

(1) M is $\mathfrak{m}$-torsion and $\operatorname{rank}_k \operatorname{soc}(M)$ is finite of rank t;

(2) M is an essential extension of a k-vector space of finite rank t;

(3) M can be embedded in a direct sum of t copies of $E_R(k)$;

(4) M is artinian.

Proof. (1) $\implies$ (2): Since M is $\mathfrak{m}$-torsion, $\operatorname{soc}(M) \subseteq M$ is essential by example A.9.

(2) $\implies$ (3): Suppose that M is essential over a k-vector space V of finite rank t. Let $e_1, \ldots, e_t$ be a k-basis for V. Then $V = \bigoplus_{i=1}^{t} ke_i$, and so

$$E_R(V) \cong E_R\left(\bigoplus_{i=1}^{t} ke_i\right) \cong \bigoplus_{i=1}^{t} E_R(k).$$

Since M is an essential extension of V, we know that $M \subseteq E(V)$, so we're done.

(3) $\implies$ (4): By Corollary A.32, we know $E_R(k)$ is artinian, therefore any finite direct sum of copies of $E_R(k)$ is artinian. Therefore any submodule of a finite direct sum of copies of $E_R(k)$ is artinian.

(4) $\implies$ (1): If $m \in M$ is any element, $Rm \supseteq \mathfrak{m}m \supseteq \mathfrak{m}^2 m \supseteq \cdots$ is a descending chain of modules, hence it stabilizes. Therefore there exists $j \geq 0$ such that $\mathfrak{m}^{j+1}m = \mathfrak{m}^j m$, and therefore $\mathfrak{m}(\mathfrak{m}^j m) = \mathfrak{m}^j m$. By Nakayama's lemma, $\mathfrak{m}^j m = 0$, so M is $\mathfrak{m}$-torsion. Since M is artinian and $\operatorname{soc}(M) \subseteq M$, $\operatorname{soc}(M)$ is artinian. But then $\operatorname{rank}_k \operatorname{soc}(M)$ is equal to the length of $\operatorname{soc}(M)$ as a k-module. Since ever element of $\operatorname{soc}(M)$ is killed by $\mathfrak{m}$, $\operatorname{soc}(M) \otimes_R k \cong \operatorname{soc}(M) \otimes_R R/\mathfrak{m} \cong \operatorname{soc}(M) \otimes_R R = \cong \operatorname{soc}(M)$, and so the length of $\operatorname{soc}(M)$ as a k-module is equal to the length of $\operatorname{soc}(M)$ as an R-module. □

Example A.34. Let $(R, \mathfrak{m}, k)$ be a DVR with $\mathfrak{m} = xR$ (e.g., $R = k[[x]]$). Then $E_R(k) \cong R_x/R$: R_x/R is divisible since if $0 \neq f \in R$ is not a unit, then $f \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$ for some j, and so $\frac{f}{x^j}$ is a unit, hence

$$f \cdot (R_x/R) = (f \cdot R_x)/R = \left(\frac{f}{x^j} R_x\right)/R = R_x/R.$$

Therefore R_x/R is an injective R-module since R is a P.I.D. Furthermore, it is (x)-torsion: if $\frac{f}{x^k} + R \in R_x/R$, then $(x)^k$ annihilates it. The socle is

$$\operatorname{soc}(R_x/R) = \left\{\frac{f}{x^k} + R \mid f \in R, \frac{xf}{x^k} \in R\right\} = \left\{\frac{a}{x} + R \mid a \in R \setminus (x)\right\}$$

and so it is generated by $\frac{1}{x}$ as a $R/(x)$-module. Therefore R_x/R is (x)-torsion and $\operatorname{rank}_k \operatorname{soc}(R_x/R) = 1$. Therefore R_x/R is an essential extension of k and $R_x/R \subseteq E_R(k)$ by Theorem A.33. Since R_x/R is injective, $R_x/R = E_R(k)$.

Theorem A.35 (Matlis Duality). *Let $(R, \mathfrak{m}, k)$ be a complete local ring, let $(-)^\vee = \operatorname{Hom}_R(-, E_R(k))$, and let M be an R-module. Then*

(1) If M is noetherian (resp. artinian) then $M^\vee$ is noetherian (resp. artinian);

(2) If M is artinian or noetherian, then the map $M \to M^{\vee\vee}$ is an isomorphism.

Remark A.36. Let M be a finitely generated module over a complete local ring $(R, \mathfrak{m}, k)$. One has isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(k, M^\vee) &\cong \operatorname{Hom}_R(k \otimes_R M, E_R(k)) \\ &\cong \operatorname{Hom}_R(M/\mathfrak{m}M, E_R(k)) \\ &\cong \operatorname{Hom}_k(M/\mathfrak{m}M, k).\end{aligned}$$

Thus the number of generators of M as an R-module is $\operatorname{rank}_k \operatorname{soc}(M^\vee)$; this number is the *type* of $M^\vee$.