A Study of The Paper "A Survey of Test Ideals" by Karl Schwede and Kevin Tucker

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Disclaimer: These notes are not guaranteed to be complete or error-free, but are desinged to be a resource for other students wishing to explore the wonderful world of positive characteristic. Please send any comments or corrections to `wdtaylor@uark.edu`. These notes are hosted at `www.wdtaylor.net/papers`.

Setting: In this paper, all rings are reduced, essentially of finite type over a field k. To be precise, $R = W^{-1}\left(\frac{k[x_1,\dots,x_n]}{I}\right)$ for some natural n, (radical) ideal $I \subseteq k[x_1,\dots,x_n]$, and multiplicatively closed subset $W \subseteq \frac{k[x_1,\dots,x_n]}{I}$.

2 Characteristic p Preliminaries

In section 2, the field k has characteristic a prime $p > 0$ (that is, $pk = 0$), and is perfect (that is, $k^p = k$).

2.1 The Frobenius Endomorphism

In characteristic $p > 0$, the Frobenius map $F : R \to R$ given by $F(r) = r^p$ is an injective ring homomorphism: we have $F(rs) = (rs)^p = r^p s^p = F(r)F(s)$; $F(x + y) = (x + y)^p = x^p + y^p = F(x) + F(y)$ since p divides any binomial coefficient $\binom{p}{k}$ with $0 < k < p$; and if $F(x) = F(y)$, then $x^p = y^p$, so $0 = x^p - y^p = (x - y)^p$ implies $x = y$ since R is reduced.

Since F is an injective endomorphism, R is naturally ring-isomorphic to its image under F. In order to distinguish the domain from the target, we can relabel the target space as $R^{1/p}$, and consider it as the "ring of p-th roots of elements of R." We will most often treat $R^{1/p}$ as an R-module, with elements labeled $r^{1/p}$ for $r \in R$ and with the action of R on $R^{1/p}$ given by $s \cdot r^{1/p} = (s^p r)^{1/p}$. In fact, for any R-module M, we define M^p to be the module with elements $m^{1/p}$ for $m \in M$, addition given by $m_1^{1/p} + m_2^{1/p} = (m_1 + m_2)^{1/p}$ and with action given by $r \cdot m^{1/p} = (r^p m)^{1/p}$. In particular, if $I = (a_1,\dots,a_d) \subseteq R$ is an ideal, then $I^{1/p} = (a_1^{1/p},\dots,a_d^{1/p}) \subseteq R^{1/p}$ (note here that this is an ideal in the ring $R^{1/p}$).

The Frobenius map induces a functor F_* on R-modules given by $F_*(M) = M^{1/p}$. In order to complete the functor definition we need to describe how F_* acts on R-module homomorphisms. Let $\varphi : M \to N$ be a homomorphism of

R-modules. We define $\varphi^{1/p} = F_*(\varphi) \in \operatorname{Hom}_R(M^{1/p}, N^{1/p})$ by $\varphi^{1/p}(m^{1/p}) = \varphi(m)^{1/p}$. We confirm that this is an R-module homomorphsim by showing it is compatible with the action of R. Let $r \in R$, then

$$\varphi^{1/p}(r \cdot m^{1/p}) = \varphi^{1/p}((r^p m)^{1/p}) = \varphi(r^p m)^{1/p} = (r^p \varphi(m))^{1/p} = r \cdot \varphi(m)^{1/p},$$

which is what we wished to show.

Exercise 2.3a. The functor F_* is exact.

We give two solutions for this exercise.

Solution 1. Let M be an R-module. Then $M^{1/p}$ is clearly an $R^{1/p}$-module under the action $r^{1/p} \cdot m^{1/p} = (r \cdot m)^{1/p}$, where the second action is the original R-action on M. Treating $M^{1/p}$ as an R-module is equivalent to simply restricting the scalars that may act on $M^{1/p}$ to the members of $R^{1/p}$ that are in the image of $F : R \to R^{1/p}$. Since restriction of scalars is an exact functor, so is F_*. □

Solution 2. We can also prove exactness directly. Let

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

be a short exact sequence of R-modules and consider the sequence obtained by applying the functor F_*:

$$0 \longrightarrow L^{1/p} \xrightarrow{\varphi^{1/p}} M^{1/p} \xrightarrow{\psi^{1/p}} N^{1/p} \longrightarrow 0\ .$$

Let $\ell^{1/p} \in \ker \varphi^{1/p}$. Then $0^{1/p} = \varphi^{1/p}(\ell^{1/p}) = \varphi(\ell)^{1/p}$, so $\varphi(\ell) = 0$. Since φ is injective, $\ell = 0$, so $\ell^{1/p} = 0^{1/p}$. Hence $\varphi^{1/p}$ is injective and the sequence is exact at $L^{1/p}$.

Let $m^{1/p} \in M$. Then $m^{1/p} \in \operatorname{im} \varphi^{1/p}$ if and only if there exists $\ell^{1/p} \in L^{1/p}$ with $m^{1/p} = \varphi^{1/p}(\ell^{1/p}) = \varphi(\ell)^{1/p}$ if and only if $m \in \operatorname{im} \varphi = \ker(\psi)$ if and only if $\psi^{1/p}(m^{1/p}) = \psi(m)^{1/p} = 0^{1/p}$, i.e. $m^{1/p} \in \ker \psi^{1/p}$. Therefore $\operatorname{im} \varphi^{1/p} = \ker \psi^{1/p}$, and the sequence is exact at $M^{1/p}$.

Let $n^{1/p} \in N^{1/p}$. Choose $m \in M$ such that $\psi(m) = n$. Then $\psi^{1/p}(m^{1/p}) = \psi(n)^{1/p} = m^{1/p}$, and so ψ is surjective and the sequence is exact at $N^{1/p}$.

Therefore the sequence is exact after applying F_*, i.e. F_* is an exact functor. □

We can iterate the functor F_* as many times as we like. We usually denote by e the number of times we iterate F_* and call the resulting (exact) functor F_*^e. For an R-module M we denote $F_*^e(M)$ by M^{1/p^e}. For the case $M = R$, we can think of R^{1/p^e} as the p^eth roots of elements in R. The R-action on M^{1/p^e} is given by $r \cdot m^{1/p^e} = (r^{p^e} m)^{1/p^e}$. Sometimes, such as when considering F-purity below, we will be able to show that it doesn't matter which e we pick when defining conditions. Other times the value of e will be very important, and we will even treat certain values as functions of e, such as when studying Hilbert-Kunz multiplicity.

Exercise 2.3b. Let $I \subseteq R$ be an ideal and $e \geq 1$. Then $(R/I)^{1/p^e}$ and $R^{1/p^e}/I^{1/p^e}$ are isomorphic as R-modules.

Solution. The sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is exact. Therefore, by the exactness of F^e_*, the sequence

$$0 \longrightarrow I^{1/p^e} \longrightarrow R^{1/p^e} \longrightarrow (R/I)^{1/p^e} \longrightarrow 0$$

is also exact. By the first isomorphism theorem, $(R/I)^{1/p^e} \cong R^{1/p^e}/I^{1/p^e}$. □

Exercise 2.1. Let $S = k[x_1, \ldots, x_d]$. Then S^{1/p^e} is a free S-module of rank p^{ed} with S-basis $\left\{(x_1^{\lambda_1} \cdots x_d^{\lambda_d})^{1/p^e}\right\}_{0 \leq \lambda_i \leq p^e - 1}$.

Solution. Let $f^{1/p^e} \in S^{1/p^e}$. For any integer d-tuple $(\lambda_1, \ldots, \lambda_d)$ with $0 \leq \lambda_i \leq p^e - 1$ for all i, denote by $f_{(\lambda_1,\ldots,\lambda_d)}$ the sum of the homogeneous parts of f where the power of x_i is congruent to λ_i modulo p^e for all i. Then f is the sum of all the $f_{(\lambda_1,\ldots,\lambda_d)}$s. Also, for each such d-tuple, $\dfrac{f_{(\lambda_1,\ldots,\lambda_d)}}{x_1^{\lambda_1} \cdots x_d^{\lambda_d}}$ is a polynomial in S with all exponents multiples of p^e, hence is a perfect p^eth power of some polynomial $g_{(\lambda_1,\ldots,\lambda_d)}$. Therefore

$$f = \sum_{0 \leq \lambda_i \leq p^e - 1} f_{(\lambda_1,\ldots,\lambda_d)} = \sum_{0 \leq \lambda_i \leq p^e - 1} (g_{(\lambda_1,\ldots,\lambda_d)})^{p^e} (x_1^{\lambda_1} \cdots x_d^{\lambda_d}), \tag{1}$$

and so

$$\begin{aligned} f^{1/p^e} &= \sum_{0 \leq \lambda_i \leq p^e - 1} \left((g_{(\lambda_1,\ldots,\lambda_d)})^{p^e} (x_1^{\lambda_1} \cdots x_d^{\lambda_d})\right)^{1/p^e} \\ &= \sum_{0 \leq \lambda_i \leq p^e - 1} g_{(\lambda_1,\ldots,\lambda_d)} \cdot (x_1^{\lambda_1} \cdots x_d^{\lambda_d})^{1/p^e}. \end{aligned}$$

Hence the set $\left\{(x_1^{\lambda_1} \cdots x_d^{\lambda_d})^{1/p^e}\right\}_{0 \leq \lambda_i \leq p^e - 1}$ generates S^{1/p^e} as an S-module.

Now suppose that $f = 0$. Then for each d-tuple $(n_1, \ldots, n_d) \in \mathbb{N}^d$, we have that the degree $(n_1, \ldots, n_d)$ component of the right hand side of equation 1 vanishes. For each i we can write $n_i = p^e q_i + \lambda_i$ with $q_i, \lambda_i \in \mathbb{N}$ and $\lambda_i < p^e$. The degree $(n_1, \ldots, n_d)$ component is then the product of the degree $(q_1, \ldots, q_d)$ component of $g_{(\lambda_1,\ldots,\lambda_d)}$ and $x_1^{\lambda_1} \cdots x_d^{\lambda_d}$. Hence the degree $(q_1, \ldots, q_d)$ component of $g_{(\lambda_1,\ldots,\lambda_d)}$ vanishes. This is true for all degrees $(n_1, \ldots, n_d) \in \mathbb{N}^d$, and hence all the polynomials $g_{(\lambda_1,\ldots,\lambda_d)}$ are zero. So the set $\left\{(x_1^{\lambda_1} \cdots x_d^{\lambda_d})^{1/p^e}\right\}_{0 \leq \lambda_i \leq p^e - 1}$ are S-linearly independent, hence an S-basis of S^{1/p^e}. □

Exercise 2.1 shows us that the functor F^e_* behaves nicely with respect to polynomial rings, giving a free module. It would be far too much to ask that this functor would always give a free module when applied to any ring, but in our setting we at least have finite generation.

Lemma 2.4. *R^{1/p^e} is a finitely generated R-module.*

Before proving Lemma 2.4 we prove an auxiliary lemma regarding the interaction of localization and F^e_*.

Lemma A. *Let M be an R-module and $W \subseteq R$ a multiplicative set. Then $W^{-1}(M^{1/p^e}) \cong (W^{-1}M)^{1/p^e}$ as $W^{-1}R$-modules.*

Proof. Let $\varphi : W^{-1}(M^{1/p^e}) \to (W^{-1}M)^{1/p^e}$ be given by $\varphi\left(\frac{m^{1/p^e}}{u}\right) = \left(\frac{m}{u^{p^e}}\right)^{1/p^e}$ and $\psi : (W^{-1}M)^{1/p^e} \to W^{-1}(M^{1/p^e})$ be given by $\psi\left(\left(\frac{m}{u}\right)^{1/p^e}\right) = \frac{(u^{p^e-1}\cdot m)^{1/p^e}}{u}$. Then φ is a well-defined $W^{-1}R$-module homomorphism:

- If $\frac{m^{1/p^e}}{u} = \frac{n^{1/p^e}}{v}$, then for some $w \in W$, $0 = w \cdot (v \cdot m^{1/p^e} - u \cdot n^{1/p^e}) = (w^{p^e}(v^{p^e}m - u^{p^e}n))^{1/p^e}$, so $\frac{m}{u^{p^e}} = \frac{n}{u^{p^e}}$, and thus
$$\varphi\left(\frac{m^{1/p^e}}{u}\right) = \left(\frac{m}{u^{p^e}}\right)^{1/p^e} = \left(\frac{n}{v^{p^e}}\right)^{1/p^e} = \varphi\left(\frac{n^{1/p^e}}{v}\right).$$
- If $\frac{r}{v} \in W^{-1}R$, then $\varphi\left(\frac{r}{v} \cdot \frac{m^{1/p^e}}{u}\right) = \frac{r}{v} \cdot \varphi\left(\frac{m^{1/p^e}}{u}\right)$ since
$$\varphi\left(\frac{r}{v} \cdot \frac{m^{1/p^e}}{u}\right) = \varphi\left(\frac{(r^{p^e} \cdot m)^{1/p^e}}{vu}\right) = \left(\frac{r^{p^e} \cdot m}{v^{p^e}u^{p^e}}\right)^{1/p^e} = \frac{r}{v} \cdot \left(\frac{m}{u^{p^e}}\right)^{1/p^e}.$$

Similarly, ψ is a well-defined $W^{-1}R$-module homomorphism:

- If $\left(\frac{m}{u}\right)^{1/p^e} = \left(\frac{n}{v}\right)^{1/p^e}$, then for some $w \in W$, $w \cdot (v \cdot m - u \cdot n) = 0$, and so
$$\begin{aligned} & w \cdot \left(v \cdot (u^{p^e-1} \cdot m)^{1/p^e} - u \cdot (v^{p^e-1} \cdot n)^{1/p^e}\right) \\ =& \left(w^{p^e}(v^{p^e}u^{p^e-1} \cdot m - u^{p^e}v^{p^e-1} \cdot n)\right)^{1/p^e} \\ =& \left(w^{p^e-1}u^{p^e-1}v^{p^e-1} \cdot (w \cdot (v \cdot m - u \cdot n))\right)^{1/p^e} \\ =& 0, \end{aligned}$$
hence
$$\psi\left(\left(\frac{m}{u}\right)^{1/p^e}\right) = \frac{(u^{p^e-1} \cdot m)^{1/p^e}}{u} = \frac{(v^{p^e-1} \cdot n)^{1/p^e}}{v} = \psi\left(\left(\frac{n}{v}\right)^{1/p^e}\right).$$

- If $\frac{r}{v} \in W^{-1}R$, then

$$\begin{aligned}
\psi\left(\frac{r}{v}\cdot\left(\frac{m}{u}\right)^{1/p^e}\right) &= \psi\left(\left(\frac{r^{p^e}\cdot m}{v^{p^e}u}\right)^{1/p^e}\right)\\
&= \frac{\left(v^{p^e(p^e-1)}u^{p^e-1}r^{p^e}\cdot m\right)^{1/p^e}}{v^{p^e}u}\\
&= \frac{v^{p^e-1}r\cdot\left(u^{p^e-1}\cdot m\right)^{1/p^e}}{v^{p^e}u}\\
&= \frac{r\cdot\left(u^{p^e-1}\cdot m\right)^{1/p^e}}{vu}\\
&= \frac{r}{v}\cdot\frac{\left(u^{p^e-1}\cdot m\right)^{1/p^e}}{u}\\
&= \frac{r}{v}\cdot\psi\left(\left(\frac{m}{u}\right)^{1/p^e}\right)
\end{aligned}$$

Now suppose that $\frac{m^{1/p^e}}{u} \in W^{-1}(M^{1/p^e})$. Then

$$\begin{aligned}
\psi\circ\varphi\left(\frac{m^{1/p^e}}{u}\right) &= \psi\left(\left(\frac{m}{u^{p^e}}\right)^{1/p^e}\right)\\
&= \frac{\left(u^{p^e(p^e-1)}\cdot m\right)^{1/p^e}}{u^{p^e}}\\
&= \frac{u^{p^e-1}\cdot m^{1/p^e}}{u^{p^e}}\\
&= \frac{m^{1/p^e}}{u}.
\end{aligned}$$

Similarly, if $\left(\frac{m}{u}\right)^{1/p^e} \in \left(W^{-1}M\right)^{1/p^e}$, then

$$\begin{aligned}
\varphi\circ\psi\left(\left(\frac{m}{u}\right)^{1/p^e}\right) &= \varphi\left(\frac{\left(u^{p^e-1}\cdot m\right)^{1/p^e}}{u}\right)\\
&= \left(\frac{u^{p^e-1}\cdot m}{u^{p^e}}\right)^{1/p^e}\\
&= \left(\frac{m}{u}\right)^{1/p^e}.
\end{aligned}$$

Therefore φ and ψ are inverses of each other, and hence $W^{-1}R$-module isomorphisms. □

We are now ready to prove lemma 2.4.

Proof of Lemma 2.4. We have that $R = W^{-1}(S/I)$ for a polynomial ring $S = k[x_1, \ldots, x_d]$, an ideal $I \subseteq S$, and a multiplicative set $W \subseteq S/I$. We know that S^{1/p^e} is finitely generated as an S-module by Exercise 2.1. Let ${\beta_i}^{1/p^e}$, $i = 1, \ldots, m$ be S-generators of S^{1/p^e}. Then $(S/I)^{1/p^e}$ is generated at an S/I-module by $(\beta_i + I)^{1/p^e}$, $i = 1, \ldots, m$: if $(f + I)^{1/p^e} \in (S/I)^{1/p^e}$, then there exist $g_i \in S$, $i = 1, \ldots, n$, such that $f^{1/p^e} = \sum_i g_i \cdot {\beta_i}^{1/p^e}$. Therefore

$$\begin{aligned}
\sum_{i=1}^{m}(g_i + I) \cdot (\beta_i + I)^{1/p^e} &= \sum_{i=1}^{m} \left(g_i^{p^e}\beta_i + I\right)^{1/p^e} \\
&= \sum_{i=1}^{m} \left(g_i^{p^e}\beta_i\right)^{1/p^e} + I^{1/p^e} \\
&= \sum_{i=1}^{m} g_i \cdot \beta_i^{1/p^e} + I^{1/p^e} \\
&= f^{1/p^e} + I.
\end{aligned}$$

Since $(S/I)^{1/p^e}$ is finitely generated as an S/I-module, $W^{-1}(S/I)^{1/p^e}$ is a finitely generated $W^{-1}(S/I)$-module. But by Lemma A, $W^{-1}(S/I)^{1/p^e} = (W^{-1}(S/I))^{1/p^e}$. So $R^{1/p^e} \cong \left(W^{-1}(S/I)\right)^{1/p^e}$ is finitely generated as an $R = W^{-1}(S/I)$-module. □

So now we ask what it is about polynomial rings that make them special with regards to the Frobenius functor? The next theorem gives us an explanation.

Theorem 2.5. *R is regular if and only if R^{1/p^e} is a locally free R-module.*

2.2 F-purity

We now examine a weaker condition than R^{1/p^e} being (locally) free. We will call a ring *F-pure* if R is a direct summand of each R^{1/p^e} (as R-modules). This is equivalent to the condition that each inclusion $R \subseteq R^{1/p^e}$ is split, i.e. that there exists an R-module homomorphism $s : R^{1/p^e} \to R$ such that $s|_R = \mathrm{id}_R$. We will call such a homomorphsim an *F-splitting of R^{1/p^e}*.

Exercise 2.7. If R is F-pure and M is an R-module, then the natural map $M \to M \otimes_R R^{1/p^e}$ is injective.

Solution. Since R is F-pure, the short exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow R^{1/p^e} \longrightarrow R^{1/p^e}/R \longrightarrow 0$$

is split exact, and hence the functor $M \otimes_R -$ preserves the exact sequence, i.e. the short exact sequence

$$0 \longrightarrow M \otimes_R R \cong M \longrightarrow M \otimes_R R^{1/p^e} \longrightarrow M \otimes_R R^{1/p^e}/R \longrightarrow 0$$

is exact. Therefore the map $M \to M \otimes_R R^{1/p^e}$ is injective. $\square$

One can use the condition of exercise 2.7 to define the concept of F-purity for more general classes of rings than we consider in this paper.

Exercise 2.8. If $R \subseteq R^{1/p^e}$ is split for some $e \geq 1$, then it is split for all $e \geq 1$

Solution. Suppose that $s : R^{1/p^e} \to R$ is an F-splitting. Then since $R \subseteq R^{1/p} \subseteq R^{1/p^e}$, restricting s to $R^{1/p}$ gives an F-splitting of $R^{1/p}$. So we can assume that $e = 1$. Now for $d \geq 1$, we have that $s^{1/p^{d-1}} : R^{1/p^d} \to R^{1/p^{d-1}}$ is a splitting of $R^{1/p^{d-1}} \subseteq R^{1/p^d}$, and so that inclusion is split. Since this applies for all $d \geq 1$,

$$R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq \cdots \subseteq R^{1/p^{d-1}} \subseteq R^{1/p^d}$$

is a composition of split inclusions, and $R \subseteq R^{1/p^d}$ is therefore split. $\square$

Exercise 2.8 lets us know that in general, to prove that R is F-pure, it suffices to show that $R \subseteq R^{1/p}$ is split.

Exercise 2.9. Suppose R is a domain. If there exists $\mathfrak{q} \in \operatorname{Spec} R$ such that $R_\mathfrak{q}$ is F-pure, then there is an open neighborhood $U \subseteq \operatorname{Spec} R$ of $\mathfrak{q}$ such that for all $\mathfrak{p} \in U$, $R_\mathfrak{p}$ is F-pure.

Solution. By Lemma 2.4, $R^{1/p}$ is a finitely generated R-module, say with generators $y_1^{1/p}, \ldots, y_n^{1/p}$. Then for any $\mathfrak{p} \in \operatorname{Spec} R$, $(R_\mathfrak{p})^{1/p} = (R^{1/p})_\mathfrak{p}$ is generated as an $R_\mathfrak{p}$-module by $\frac{y_1^{1/p}}{1}, \ldots, \frac{y_n^{1/p}}{1}$. Since $R_\mathfrak{q}$ is F-pure, there exists a splitting $s : (R^{1/p})_\mathfrak{q} \to R_\mathfrak{q}$. For each i, let $x_i \in R$, $v_i \in R \setminus \mathfrak{q}$ such that $s\left(\dfrac{y_i^{1/p}}{1}\right) = \dfrac{x_i}{v_i}$.

Now let $U = \{\mathfrak{p} \in \operatorname{Spec} R | \forall i, v_i \notin \mathfrak{p}\} = \operatorname{Spec} R \setminus \bigcup_{i=1}^{n} V(v_i)$. Then U is open since it is the complement of a finite union of closed sets. Furthermore, $\mathfrak{q} \in U$ since $v_i \notin \mathfrak{q}$ for all i, showing that U is a neighborhood of $\mathfrak{q}$. Now let $\mathfrak{p} \in U$ and define a function $t : (R^{1/p})_\mathfrak{p} \to R_\mathfrak{p}$ by setting $t\left(\dfrac{y_i^{1/p}}{1}\right) = \dfrac{x_i}{v_i}$ and extending $R_\mathfrak{p}$-linearly. We need to show that t is well-defined. Suppose that $r_i \in R$, $u_i \in R \setminus \mathfrak{p}$ such that $\sum_i \frac{r_i}{u_i} \cdot \frac{y_i^{1/p}}{1} = 0^{1/p}$. Let $u = u_1 \cdots u_n$, and for all i, let $r_i' = \frac{ur_i}{u_i} \in R$. Then

$$0^{1/p} = \sum_i \frac{r_i}{u_i} \cdot \frac{y_i^{1/p}}{1} = \sum_i \frac{r_i'}{u} \cdot \frac{y_i^{1/p}}{1} = \frac{\sum_i r_i' \cdot y_i^{1/p}}{u},$$

which implies, since $R^{1/p}$ is a domain, that $\sum_{i=1}^n r_i' \cdot y_i^{1/p} = 0^{1/p}$. Therefore $\sum_{i=1}^n \frac{r_i'}{1} \cdot \frac{y_i^{1/p}}{1} = 0^{1/p}$ in any localization of $R^{1/p}$, and so in $R_\mathfrak{q}$,

$$0 = s(0^{1/p}) = s\left(\sum_{i=1}^n \frac{r_i'}{1} \cdot \frac{y_i^{1/p}}{1}\right) = \sum_{i=1}^n \frac{r_i'}{1} \cdot \frac{x_i}{v_i}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n}\frac{r_i}{u_i}\cdot t\left(\frac{y_i^{1/p}}{1}\right) &= \sum_{i=1}^{n}\frac{r_i}{u_i}\cdot\frac{x_i}{v_i}\\ &= \frac{1}{u}\sum_{i=1}^{n}\frac{r_i'}{1}\cdot\frac{x_i}{v_i}\\ &= 0.\end{aligned}$$

So t is well-defined. Now let $r_i \in R$ such that $\sum_{i=1}^{n} r_i \cdot y_i^{1/p} = 1^{1/p}$. Then

$$\frac{1}{1} = s(1^{1/p}) = s\left(\sum_{i=1}^{n}\frac{r_i}{1}\cdot\frac{y_i^{1/p}}{1}\right) = \sum_{i=1}^{n}\frac{r_i}{1}\cdot\frac{x_i}{v_i} = \frac{\sum_{i=1}^{n} r_i' x_i}{v},$$

where $v = v_1 \cdots v_n$ and $r_i' = \frac{v r_i}{v_i}$. Hence $\sum_{i=1}^{n} r_i' x_i = v$. So

$$\begin{aligned} t\left(\frac{1^{1/p}}{1}\right) &= t\left(\sum_{i=1}^{n}\frac{r_i}{1}\cdot\frac{y_i^{1/p}}{1}\right)\\ &= \sum_{i=1}^{n}\frac{r_i}{1}\cdot t\left(\frac{y_i^{1/p}}{1}\right)\\ &= \left(\sum_{i=1}^{n}\frac{r_i}{1}\cdot\frac{x_i}{v_i}\right)\\ &= \frac{\sum_{i=1}^{n} r_i' x_i}{v}\\ &= \frac{1}{1}\end{aligned}$$

Therefore t is an F-splitting for $R_{\mathfrak{p}}$. This holds for all primes $\mathfrak{p}$ in $U_v = \operatorname{Spec} R \setminus \bigcup_i V(v_i)$, which is an open set in $\operatorname{Spec} R$ containing $\mathfrak{q}$. □

In a similar vein, we can use the F-purity of localizations of R to conclude facts about the F-purity of R itself.

Exercise 2.10. If $R_{\mathfrak{m}}$ is F-pure for each maximal ideal $\mathfrak{m}$ of R, then R is F-pure.

We prove an auxiliary lemma before solving exercise 2.10.

Lemma B. $R \subseteq R^{1/p^e}$ splits if and only if the "evaluation at 1" homomorphsim $\varphi : \operatorname{Hom}_R(R^{1/p^e}, R) \to R$ is surjective.

Proof. By exercise 2.8, we can assume $e = 1$. If $R \subseteq R^{1/p}$ splits, then there exists a splitting $s \in \operatorname{Hom}_R(R^{1/p}, R)$. So $s(1^{1/p}) = 1$, and therefore for any $r \in R$, $r \cdot s \in \operatorname{Hom}_R(R^{1/p^e}, R)$ and

$$\varphi(r \cdot s) = (r \cdot s)(1^{1/p}) = r \cdot s(1^{1/p}) = r.$$

So φ is surjective.

If φ is surjective, then there exists $s \in \operatorname{Hom}_R(R^{1/p^e}, R)$ such that $s(1^{1/p}) = \varphi(s) = 1$. Therefore s is an F-splitting for $R^{1/p}$, since for any $r \in R$, the image of r under the inclusion $R \subseteq R^{1/p}$ is $(r^p)^{1/p}$, and $s((r^p)^{1/p}) = s(r \cdot 1^{1/p}) = r \cdot s(1^{1/p}) = r$. $\square$

Solution to Exercise 2.10. Let $\varphi : \operatorname{Hom}_R(R^{1/p^e}, R) \to R$ be the evaluation at 1 homomorphism. By lemma B, for each maximal ideal, the function $\varphi_{\mathfrak{m}}$: $(\operatorname{Hom}_R(R^{1/p^e}, R))_{\mathfrak{m}} \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}^{1/p^e}, R_{\mathfrak{m}}) \to R_{\mathfrak{m}}$ is surjective. Therefore φ is surjective, and so R is F-pure by lemma B. $\square$

Fedder's Criterion below is a great test for F-purity in some siutations. To prepare for the statement we need a few facts about how ideals interact with Frobenius. By $I^{[p^e]}$ we mean the ideal generated by the p^eth powers of elements in I; to be precise if $I = (a_1, \ldots, a_d)$, then $I^{[p^e]} = (a_1^{p^e}, \ldots, a_d^{p^e})_R$.

Exercise 2.12. $(I^{[p^e]})^{1/p^e} = I \cdot R^{1/p^e}$

Solution. Suppose $I = (a_1, \ldots, a_d)$. Then

$$\begin{aligned} I \cdot R^{1/p^e} &= (a_1, \ldots, a_d) \cdot R^{1/p^e} \\ &= \sum_{i=1}^{d} a_i \cdot R^{1/p^e} \\ &= \sum_{i=1}^{d} (a_i^{p^e})^{1/p^e} R^{1/p^e} \\ &= \left(\sum_{i=1}^{d} a_i^{p^e} R \right)^{1/p^e} \\ &= \left((a_1^{p^e}, \ldots, a_d^{p^e}) R \right)^{1/p^e} \\ &= (I^{[p^e]})^{1/p^e}. \end{aligned}$$

$\square$

Exercise 2.12 proves that $I^{[p^e]}$ is independent of the generators chosen for I, a fact that is not immediately apparent.

Exercise 2.13. Suppose that R is a regular local ring and $I \subseteq R$ is an ideal. If $x \in R$, then $x \in I^{[p^e]}$ if and only if $\phi(x^{1/p^e}) \in I$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$.

Solution. Suppose $I = (a_1, \ldots, a_d)$. For any $r_i \in R$, and any $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$,

$$\phi\left(\left(\sum_{i=1}^{d} r_i a_i^{p^e} \right)^{1/p^e} \right) = \phi\left(\sum_{i=1}^{d} a_i \cdot r_i^{1/p^e} \right) = \sum_{i=1}^{d} a_i \cdot \phi(r_i^{1/p^e}) \in I,$$

which proves the forward direction. Now suppose that $x \in R$ such that $\phi(x^{1/p^e}) \in I$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Since R is regular local, Theorem 2.5 gives us that R^{1/p^e} is a (finitely generated) free R-module. Let $e_1^{1/p^e}, \ldots, e_n^{1/p^e}$ be an R-basis for R^{1/p^e}. Then there exist $r_i \in R$ such that $x^{1/p^e} = r_1 \cdot e_1^{1/p^e} + \cdots + r_n \cdot e_n^{1/p^e}$. For each generator e_i the projection map π_i taking e_i^{1/p^e} to 1 and e_j^{1/p^e} to 0 for $j \neq i$ is an R-module homomorphsim. Hence $r_i = \pi_i(x^{1/p^e}) \in I$. Therefore

$$x^{1/p^e} = r_1 \cdot e_1^{1/p^e} + \cdots + r_n \cdot e_n^{1/p^e} \in I \cdot R^{1/p^e} = \left(I^{[p^e]}\right)^{1/p^e}$$

which proves that $x \in I^{[p^e]}$. $\square$

The next theorem provides one of our most powerful tools in determining F-purity.

Theorem 2.14. (Fedder's Criterion) *Suppose that $S = k[x_1, \ldots, x_n]$ and $R = S/I$ for some radical ideal $I \subseteq S$. Then for any $\mathfrak{q} \in \operatorname{Spec} R \subseteq \operatorname{Spec} S$, $R_\mathfrak{q}$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{q}^{[p]}$.*

The following exercises demonstrate the use of Fedder's Criterion.

Exercise 2.16. The ring $R = \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$ is not F-pure for $p = 2, 3, 5, 11$, and in general if $p \not\equiv 1 \mod 3$, and is F-pure for $p = 7, 13$, and in general if $p \equiv 1 \mod 3$.

Solution. The algebraic variety $V(x^3 + y^3 + z^3)$ has one singular point at the origin, since that is the only place where the partials of $x^3 + y^3 + z^3$ vanish. Therefore, for any maximal ideal $\mathfrak{m} \neq (x, y, z)$, the ring $R_\mathfrak{m}$ is regular, hence F-pure. So now let $\mathfrak{m} = (x, y, z)$, whence $\mathfrak{m}^{[p]} = (x^p, y^p, z^p)$. Since $\mathfrak{m}^{[p]}$ is a monomial ideal, a polynomial f belongs to $\mathfrak{m}^{[p]}$ if and only if each monomial term of f belongs to $\mathfrak{m}^{[p]}$. Note that $(I^{[p]} : I) = I^{[p-1]} = \left((x^3 + y^3 + z^3)^{p-1}\right)$.

If $p \equiv 1 \mod 3$, then $p-1$ is divisible by 3, and one of the terms of $(x^3+y^3+z^3)^{p-1}$ is $x^{3 \cdot \frac{p-1}{3}} y^{3 \cdot \frac{p-1}{3}} z^{3 \cdot \frac{p-1}{3}} = x^{p-1}y^{p-1}z^{p-1} \notin \mathfrak{m}^{[p]}$. Therefore $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$, and so $R_\mathfrak{m}$ is F-pure. By Exercise 2.10, R is F-pure.

Suppose $p \not\equiv 1 \mod 3$. Every term of $(x^3 + y^3 + z^3)^{p-1}$ is of the form $x^{3e_1}y^{3e_2}z^{3e_3}$, where $e_1, e_2, e_3 \in \mathbb{N}$ and $e_1 + e_2 + e_3 = p - 1$. Since $p - 1$ is not divisible by 3, $e_i > \frac{p-1}{3}$ for some i. If $e_1 > \frac{p-1}{3}$, then $3e_1 > p - 1$, which means $3e_1 \geq p$. But this means that $x^{3e_1}y^{3e_2}z^{3e_3} \in (x^p, y^p, z^p) = \mathfrak{m}^{[p]}$. Similarly, if $e_2 > \frac{p-1}{3}$ or $e_3 > \frac{p-1}{3}$, then $x^{3e_1}y^{3e_2}z^{3e_3} \in (x^p, y^p, z^p) = \mathfrak{m}^{[p]}$. Therefore $\left((x^3 + y^3 + z^3)^{p-1}\right) \subseteq \mathfrak{m}^{[p]}$, and so $R_\mathfrak{m}$ is not F-pure. Hence R is not F-pure. $\square$

Exercise 2.17. Let $S = \mathbb{F}_p[x_1, \ldots, x_n]$, $f = xy - z^2 \in S$, $g = x^4 + y^4 + z^4 \in S$. Then for all p, $S/(f)$ is F-pure and $S/(g)$ is not F-pure.

Solution. The varieties $S/(f)$ and $S/(g)$ are singular only at the origin, so by an argument similar to the beginning of the previous exercise, we need only check

Fedder's Criterion at the maximal ideal $\mathfrak{m} = (x, y, z)$, which has bracket power $\mathfrak{m}^{[p]} = (x^p, y^p, z^p)$.

If $I = (f)$, then $I^{[p]} = (f^p)$ and $(I^{[p]} : I) = (f^{p-1})$. But the polynomial $f^{p-1} = (xy - z^2)^{p-1}$ contains the term $x^{p-1}y^{p-1} \notin \mathfrak{m}^{[p]}$, so $f^{p-1} \notin \mathfrak{m}^{[p]}$. Therefore $S/(f)$ is F-pure.

If $I = (g)$, then $I^{[p]} = (g^p)$ and $(I^{[p]} : I) = (g^{p-1})$. Each term of $(x^4 + y^4 + z^4)^{p-1}$ is of the form $x^{4e_1}y^{4e_2}z^{4e_3}$ for some $e_1, e_2, e_3 \in \mathbb{N}$ with $e_1 + e_2 + e_3 = p - 1$. Therefore $e_i \geq \frac{p-1}{3}$ for some i. Then $4e_i \geq \frac{4}{3}(p-1) > p - 1$, and so $4e_i \geq p$. Hence $x^{4e_1}y^{4e_2}z^{4e_3} \in \mathfrak{m}^{[p]}$, and so $(x^4 + y^4 + z^4)^{p-1} \in \mathfrak{m}^{[p]}$. Therefore $(I^{[p]} : I) \subseteq \mathfrak{m}^{[p]}$, and so $(S/(g))_{\mathfrak{m}}$ is not F-pure. Hence $S/(g)$ is not F-pure. $\square$

Exercise 2.18. For any F-pure ring R with fraction field K, if $x \in K$ such that $x^p \in R$, then $x \in R$.

Solution. The fraction field K is the ring R localized at the set W of nonzero elements. Let $\varphi : R^{1/p} \to R$ be an F-splitting of R. Then $W^{-1}\varphi : W^{-1}R^{1/p} \to W^{-1}R$ is a splitting extending φ, and since $W^{-1}R = K$ and $W^{-1}R^{1/p} = (W^{-1}R)^{1/p} = K^{1/p}$, $W^{-1}\varphi$ (which we will now write as ϕ) is a splitting for K. Now let $x \in K$ such that $x^p \in R$. Then

$$x = \phi((x^p)^{1/p}) \in \phi(R^{1/p}) = \varphi(R^{1/p}) = R. \qquad \square$$