A Study of The Paper "A Survey of Test Ideals" by Karl Schwede and Kevin Tucker

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3 The Test Ideal

In this section, all rings are domains essentially finite type over a perfect field k of characteristic $p > 0$.

3.1 Test ideals of map-pairs

We begin by defining an ideal that caputes information about the ring R and an R-module homomorphism. For $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, let $\tau(R, \phi)$ be the smallest nonzero ideal J such that $\phi(J^{1/p^e}) \subseteq J$. It is nonobvious that such an ideal exists, but we will have a method of constructing it. If an ideal $I \subseteq R$ is such that $\phi(I^{1/p^e}) \subseteq I$, we say that I is ϕ-compatible.

Exercise 3.3. $\phi(\tau(R, \phi)^{1/p^e}) = \tau(R, \phi)$.

Solution. By definition, $\tau(R, \phi)$ is ϕ-compatible, so $\phi(\tau(R, \phi)^{1/p^e}) \subseteq \tau(R, \phi)$. Applying F^e_* and then ϕ to both sides, we get that

$$\phi((\phi(\tau(R, \phi)^{1/p^e}))^{1/p^e}) \subseteq \phi(\tau(R, \phi)^{1/p^e}),$$

proving that $\phi(\tau(R, \phi)^{1/p^e})$ is ϕ-compatible. Hence $\tau(R, \phi) \subseteq \phi(\tau(R, \phi)^{1/p^e})$, and so the two are equal. □

Exercise 3.4. Suppose that $\phi : R^{1/p^e} \to R$ is surjective. Then $\tau(R, \phi)$ is radical and $R/\tau(R, \phi)$ is F-pure.

Solution. Suppose $x \in R$ such that $x^{p^e} \in \tau(R, \phi)$. Then since ϕ is surjective,

$$x \in x \cdot \phi\left(R^{1/p^e}\right) = \phi\left(x \cdot R^{1/p^e}\right) = \phi\left((x^{p^e} R)^{1/p^e}\right) \subseteq \phi\left(\tau(R, \phi)^{1/p^e}\right) = \tau(R, \phi).$$

Now for any $n \geq 1$, suppose $x^n \in \tau(R, \phi)$. Then $n < p^{ec}$ for some $c \in \mathbb{N}$, and so $x^{p^{ec}} \in \tau(R, \phi)$, which implies that $x^{p^{e(c-1)}} \in \tau(R, \phi)$, and proceeding for c steps, $x = x^{p^{e \cdot 0}} \in \tau(R, \phi)$, and so $\tau(R, \phi)$ is radical.

Let $d \in R$ such that $\phi(d^{1/p^e}) = 1$. Let $\psi : R^{1/p^e} \to R/\tau(R, \phi)$ be given by $\phi(r^{1/p^e}) = \phi((dr)^{1/p^e}) + \tau(R, \phi)$. Since $\phi((d\tau(R, \phi))^{1/p^e}) \subseteq \tau(R, \phi)$, we have that $\tau(R, \phi)^{1/p^e} \subseteq \ker \psi$. Therefore we can construct $\bar{\psi} : R^{1/p^e}/\tau(R, \phi)^{1/p^e} \to R/\tau(R, \phi)$ given by $\bar{\psi}(r^{1/p^e} + \tau(R, \phi)^{1/p^e}) = \psi(r^{1/p^e})$. Now

$$\bar{\psi}(1^{1/p^e} + \tau(R, \phi)^{1/p^e}) = \psi(1^{1/p^e}) = \phi(d^{1/p^e}) = 1 + \tau(R, \phi).$$

Therfore $\bar{\psi}$ is an F-splitting for $R/\tau(R, \phi)$. □

Exercise 3.5. Suppose that $R = \mathbb{F}_2[x, y]$, then $R^{1/2} = R \oplus R \cdot x^{1/2} \oplus R \cdot y^{1/2} \oplus R \cdot (xy)^{1/2}$.

Part a) Let $\alpha : R^{1/2} \to R$ be given by $\alpha((xy)^{1/2}) = 1$ and $\alpha(1) = \alpha(x^{1/2}) = \alpha(y^{1/2}) = 0$. Then $\tau(R, \alpha) = R$.

Solution. For a monomial $x^m y^n \in R$, $\alpha(x^m y^n)$ is $x^{(m-1)/2} y^{(n-1)/2}$ (with degree $\frac{m+n}{2} - 1$) if m and n are odd and 0 otherwise. In particular, if $f \neq 0$ has degree d, then $\alpha(f^{1/2})$ is either zero or nonzero of degree less than or equal to $\frac{d}{2} - 1 < d$. Let J be a nonzero ideal of R and let $f \in J$ be a nonzero polynomial in J of minimal degree d.

If $\alpha(f^{1/2}) \neq 0$, then $\alpha(f^{1/2})$ is of degree less than the degree of f, so is not in J, hence J is not α-compatible.

If $\alpha(f^{1/2}) = 0$, then all terms of f are of the form $x^m y^n$ with m and n not both odd. Therefore either xf, yf, or xyf has a term with the powers on x and y both odd, call such a multiple g. Then $g \in J$, $\alpha(g^{1/2}) \neq 0$, and the degree of g is at most $d + 2$. Therefore $\alpha(g^{1/2})$ has degree at most $\frac{d+2}{2} - 1 = \frac{d}{2}$. If $d > 0$, then $\frac{d}{2} < d$ and so $\alpha(g^{1/2}) \notin J$, so J is not α-compatible. If $d = 0$, then f is a constant and $J = R$. Therefore the only nonzero α-compatible ideal of R is R, and so $\tau(R, \phi) = R$. □

Part b) Let $\beta : R^{1/2} \to R$ be given by $\beta(x^{1/2}) = 1$ and $\beta(1) = \beta(y^{1/2}) = \beta((xy)^{1/2}) = 0$. Then $\tau(R, \beta) = (y)$.

Solution. For a monomial $x^m y^n \in R$, $\beta(x^m y^n)$ is $x^{(m-1)/2} y^{n/2}$ (with degree $\frac{m+n-1}{2}$) if m is odd and n is even and 0 otherwise. In particular, if $f \neq 0$ has degree d, then $\beta(f^{1/2})$ is either zero or nonzero of degree at most $\frac{d-1}{2} < d$. Let J be a nonzero ideal of R and let $f \in J$ be a nonzero polynomial in J of minimal degree d.

If $\beta(f^{1/2}) \neq 0$, then $\beta(f^{1/2})$ is of degree less than the degree of f, so is not in J, hence J is not β-compatible.

If $\beta(f^{1/2}) = 0$, then all terms of f are of the form $x^m y^n$ with either m even or n odd. Therefore either xf, yf, or xyf has a term with the power on x odd and the power on y even.

Suppose that xf or yf has some term with the power on x odd and the power on y even and call this multiple g. Then $g \in J$, $\beta(g^{1/2}) \neq 0$, and the degree of g is $d+1$. Therefore $\beta(g^{1/2})$ has degree at most $\frac{d+1-1}{2} = \frac{d}{2}$. If $d > 0$, then $\frac{d}{2} < d$ and so $\beta(g^{1/2}) \notin J$, so J is not β-compatible. If $d = 0$, then f is a constant and $J = R$.

Now suppose that neither xf or yf has some term with the power on x odd and the power on y even. Then every term of f must have even power on x and odd power on y. In this case xyf has all terms with the power on x even and the power on y odd, call this multiple g. Then the degree of g is $d+2$. Then $\beta(g^{1/2})$ has degree at most $\frac{d+2-1}{2} = \frac{d+1}{2}$. If $d > 1$, then $\frac{d+1}{2} < d$, and so $\beta(g^{1/2}) \notin J$, so J is not β-compatible. If $d = 1$, then since every power of f must have even power on x and odd power on y we must have that $f = y$. Therefore if J is a proper β-compatible ideal then J must contain y.

We finish by showing that (y) is β-compatible. Suppose $f \in (y)$. Then the terms of $f^{1/2}$ that are not killed by β are the ones that have odd degree in x and the even degree at least 2 in y, call the sum of the terms of this form g. Then we can write $g = y^2 h$ for some $h \in \mathbb{F}_2[x, y]$. So $\beta(f^{1/2}) = \beta((y^2h)^{1/2}) = \beta(y \cdot h^{1/2}) = y\beta(h^{1/2}) \in (y)$. So (y) is β-compatible.

Therefore $\tau(R, \beta) = (y)$. □

Part c) Let $\gamma : R^{1/2} \to R$ be given by $\gamma(1^{1/2}) = 1$ and $\beta(x^{1/2}) = \beta(y^{1/2}) = \beta((xy)^{1/2}) = 0$. Then $\tau(R, \gamma) = (xy)$.

Solution. For a monomial $x^m y^n \in R$, $\gamma(x^m y^n)$ is $x^{m/2} y^{n/2}$ (with degree $\frac{m+n}{2}$) if m and n are even and 0 otherwise. In particular, if $f \neq 0$ has degree d, then $\gamma(f^{1/2})$ is either zero or nonzero of degree at most $\frac{d}{2}$. Let J be a nonzero ideal of R and let $f \in J$ be a nonzero polynomial in J of minimal degree d.

If $\gamma(f^{1/2}) \neq 0$, then $\gamma(f^{1/2})$ is of degree $\frac{d}{2}$. If $d > 0$, then this is less than the degree of f, so $\gamma(f^{1/2})$ is not in J, hence J is not γ-compatible. If $d = 0$, then f is a constant, so $J = R$.

If $\gamma(f^{1/2}) = 0$, then all terms of f are of the form $x^m y^n$ with m and n not both even. Therefore either xf, yf, or xyf has a term with the powers on x and y both even.

Suppose that xf or yf has some term with the powers on x and y both even and call this multiple g. Then $g \in J$, $\gamma(g^{1/2}) \neq 0$, and the degree of g is $d+1$. Therefore $\gamma(g^{1/2})$ has degree at most $\frac{d+1}{2}$. If $d > 1$, then $\frac{d}{2} < d$ and so $\gamma(g^{1/2}) \notin J$, so J is not γ-compatible. If $d = 1$, then $f = x$ or $f = y$, and so $xy \in J$. The case $d = 0$ is impossible.

Now suppose that neither xf or yf has some term with the powers on x and y both even. Then every term of f must have odd powers on x and y. In this case xyf has all terms with the powers on x and y both even, call this multiple g. Then the degree of g is $d+2$, and $\gamma(g^{1/2})$ has degree at most (in fact exactly) $\frac{d+2}{2}$. If $d > 2$, then $\frac{d+1}{2} < d$, and so $\gamma(g^{1/2}) \notin J$, so J is not β-compatible. The cases $d = 0$ and $d = 1$ are impossible. If $d = 2$, then since every power of f must have odd powers on x and y we must have that $f = xy$. Therefore if J is a proper γ-compatible ideal then J must contain xy.

We finish by showing that (xy) is γ-compatible. Suppose $f \in (xy)$. Then the terms of $f^{1/2}$ that are not killed by γ are the ones that have even degrees at least 2 in x and y, call the sum of the terms of this form g. Then we can write $g = x^2y^2h$ for some $h \in \mathbb{F}_2[x,y]$. So $\gamma(f^{1/2}) = \gamma((x^2y^2h)^{1/2}) = \gamma(xy \cdot h^{1/2}) = xy\gamma(h^{1/2}) \in (xy)$. So (xy) is γ-compatible. $\square$

The question of whether such an ideal $\tau(R,\phi)$ exists is answered in the affirmative by Lemma 3.6 and Theorem 3.8.

Lemma 3.6. Suppose that $\phi : R^{1/p} \to R$ is a nonzero R-module homomorphism. Then there exists nonzero $c \in R$ such that for all nonzero $d \in R$, there exists $n > 0$ such that $c \in \phi^n\left((dR)^{1/p^{ne}}\right)$.

In the above lemma, ϕ^n is the composition

$$R^{1/p^{ne}} \xrightarrow{\phi^{1/p^{(n-1)e}}} R^{1/p^{(n-1)e}} \xrightarrow{\phi^{1/p^{(n-2)e}}} \cdots \xrightarrow{\phi^{1/p^{e}}} R^{1/p^{e}} \xrightarrow{\phi} R.$$

We call an element c satisfying the conditions of Lemma 3.6 a *test element* for ϕ.

Theorem 3.8. Fix any $c \in R$ a test element for ϕ. Then

$$\tau(R,\phi) = \sum_{e \geq 0} \phi^n\left((cR)^{1/p^{ne}}\right)$$

Here, by ϕ^0 we mean the identity map $R \to R$.

Proof. Let $T = \sum_{e\geq 0} \phi^n\left((cR)^{1/p^{ne}}\right)$. Now $\phi(T^{1/p}) = \sum_{e\geq 1} \phi^n\left((cR)^{1/p^{ne}}\right)$, so T is ϕ-compatible. If I is any ϕ-compatible ideal, then there exists a nonzero $d \in I$, and so since c satisfies the condition of Lemma 3.6, there exists $n \in \mathbb{N}$ such that $c \in \phi^n((dR)^{1/p^{ne}})$. But also,

$$\phi^n\left((dR)^{1/p^{ne}}\right) \subseteq \phi^n\left(I^{1/p^{ne}}\right) \subseteq \phi^{n-1}\left(I^{1/p^{(n-1)e}}\right) \subseteq \cdots \subseteq \phi\left(I^{1/p^{e}}\right) \subseteq I,$$

and so $c \in I$. But then for any $n \in \mathbb{N}$, $\phi^n\left((cR)^{p^{ne}}\right) \subseteq \phi^n\left(I^{1/p^{ne}}\right) \subseteq I$ as above. Therefore $T \subseteq I$. Therefore T is the smallest ϕ-compatible ideal of R, i.e. $T = \tau(R,\phi)$. $\square$

Exercise 3.9. $\tau(R,\phi) = \tau(R,\phi^m)$ for any $m > 0$.

Solution. Let $c \in R$ be a test element for ϕ^m. Let $0 \neq d \in R$, and choose $n > 0$ such that $c \in (\phi^m)^n\left((dR)^{1/p^{mne}}\right) = \phi^{mn}\left((dR)^{1/p^{mne}}\right)$. Therefore c is a test element for ϕ.

We have that

$$\tau(R,\phi) = \sum_{e\geq 0} \phi^n\left((cR)^{1/p^{ne}}\right) \text{ and } \tau(R,\phi^m) = \sum_{e\geq 0} \phi^{mn}\left((cR)^{1/p^{mne}}\right).$$

Since every term in the sum for $\tau(R,\phi^m)$ is also a term of the sum for $\tau(R,\phi)$, we have that $\tau(R,\phi^m) \subseteq \tau(R,\phi)$.

We claim that for all n, $\phi^n((cR)^{1/p^{ne}})$ contains a nonzero element. Let $k \in \mathbb{N}$ such that $c \in \phi^k((cR)^{1/p^e})$. Then

$$cR \subseteq \phi^k((cR)^{1/p^e}) \subseteq \phi^{2k}((cR)^{1/p^{2k}}) \subseteq \cdots,$$

so for any $j \in \mathbb{N}$, $c \in \phi^{kj}((cR)^{1/p^{kje}})$ and so $\phi^{kj}((cR)^{1/p^{kje}}) \neq 0$. Now for any $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $kj > n$, and so since

$$0 \neq \phi^{kj}((cR)^{1/p^{kje}} = \phi^{kj-n}\left((\phi^n)^{1/p^{(kj-n)e}}((cR)^{1/p^{ne}})\right),$$

we must have that $\phi^n((cR)^{1/p^{ne}}) \neq 0$.

Therefore we can pick $0 \neq d_j \in \phi^j((cR)^{/1p^{je}})$ for $j = 1, 2, \ldots, m$. Let $d = \prod_i d_i$. Then $d \in \phi^j((cR)^{/1p^{je}})$ for $j = 1, 2, \ldots, m$. Let $n \in \mathbb{N}$, and choose $k \in \mathbb{N}$ such that $c \in \phi^k((dR)^{1/p^{ke}}$. Then pick j in $\{1, \ldots, m\}$ such that $n + k + j \equiv 0 \mod m$. Now

$$\phi^n((cR)^{1/p^{ne}}) \subseteq \phi^{n+k}((dR)^{1/p^{(n_k)e}}) \subseteq \phi^{n+k+j}((cR)^{1/p^{(n+k+j)e}},$$

but the term on the right is a summand of $\tau(R,\phi^m)$. Therefore $\phi^n((cR)^{1/p^{ne}}) \subseteq \tau(R,\phi^m)$, and since n was arbitrary, this shows that $\tau(R,\phi) \subseteq \tau(R,\phi^m)$. □

Exercise 3.10. If W is a multiplicative system of R and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, then $\tau(W^{-1}R, W^{-1}\phi) = W^{-1}\tau(R,\phi)$.

Solution. Let $c \in R$ be a test element for ϕ. Let $0 \neq \frac{d}{u} \in (W^{-1}R)$. Then $d \neq 0$, and so there exists $n \in \mathbb{N}$ such that $c \in \phi^n((dR)^{1/p^{ne}})$. Then

$$\begin{aligned}
&(W^{-1}\phi)^n\left(\left(\frac{d}{u} \cdot W^{-1}R\right)^{1/p^{ne}}\right) \\
=&(W^{-1}\phi)^n\left(\left(W^{-1}(dR)\right)^{1/p^{ne}}\right) \\
=&(W^{-1}\phi)^n\left(W^{-1}\,(dR)^{1/p^{ne}}\right) \\
=&W^{-1}\phi^n\left((dR)^{1/p^{ne}}\right) \\
\ni&\frac{c}{1}.
\end{aligned}$$

Therefore $\frac{c}{1}$ is a test element for W^{-1}. Hence,

$$\begin{aligned}
\tau(W^{-1}R, W^{-1}\phi) &= \sum_{n\geq 0} (W^{-1}\phi)^n \left(\left(\frac{c}{1} \cdot W^{-1}R \right)^{1/p^{ne}} \right) \\
&= \sum_{n\geq 0} (W^{-1}\phi)^n \left(W^{-1} (cR)^{1/p^{ne}} \right) \\
&= \sum_{n\geq 0} W^{-1}\phi^n \left((cR)^{1/p^{ne}} \right) \\
&= W^{-1} \sum_{n\geq 0} \phi^n \left((cR)^{1/p^{ne}} \right) \\
&= W^{-1}\tau(R, \phi).
\end{aligned}$$

Exercise 3.11. Let c be a test element for $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, let $J_0 = cR$ and for $n \geq 1$, $J_n = J_{n-1} + \phi(J_{n-1}^{1/p^e})$. Then for $n \gg 0$, $J_n = \tau(R, \phi)$.

Solution. Note that the chain of ideals $J_0 \subseteq J_1 \subseteq \cdots$ is increasing. Therefore, since R is noetherian, $J_n = J_{n+1} = \cdots$ for some $n > 0$. Therefore $J_n = J_n + \phi({J_n}^{1/p^e})$ and therefore $\phi(J_n^{1/p^e}) \subseteq J_n$, i.e. J_n is ϕ-compatible. Hence $J_n \supseteq \tau(R, \phi)$.

Now $J_0 = cR \subseteq \tau(R, \phi)$, and $\tau(R, \phi)$ is ϕ-compatible. If $J_k \subseteq \tau(R, \phi)$, then

$$J_{k+1} = J_k + \phi(J_k^{1/p^e}) \subseteq \tau(R, \phi) + \phi(\tau(R, \phi)^{1/p^e}) = \tau(R, \phi).$$

Therefore, by induction, $J_n \subseteq \tau(R, \phi)$, and so $J_n = \tau(R, \phi)$. □

3.2 Test ideals of rings

We can define an ideal depending only on the ring structure of R by simultaineously considering all homomorphisms $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. To be precise, we define the *test ideal of* R, denoted $\tau(R)$, to be the smallest nonzero ideal J such that $J \subseteq \phi(J^{1/p^e})$ for all $e \geq 0$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$.

Exercise 3.13. If $S = k[x_1, \ldots, x_n]$, then $\tau(S) = S$.

Solution. We will show that if J is any nonzero ideal of S then there exists $e \geq 0$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ such that $\phi(J^{1/p^e}) = S$, which will prove the statement. Let J be a nonzero ideal of S, and let $0 \neq f \in J$. Choose $e > 0$ such that p^e is greater than the highest power of any single variable that appears in f. By exercise 2.1, S^{1/p^e} is a free S-module with basis $x_1^{\lambda_1/p^e} \cdots x_n^{\lambda_n/p^e}$ for $0 \leq \lambda_i \leq p^e - 1$. Then the monomials of f^{1/p^e} are S-linearly independent in S^{1/p^e}. Choose an exponenet vector $(\ell_1, \ldots, \ell_n)$ such that the coefficient of $x_1^{\ell_1} \cdots x_n^{\ell_n}$ is $a_\ell \neq 0$. Let $\phi : S^{1/p^e} \to S$ be given by $\phi(x_1^{\ell_1/p^e} \cdots x_n^{\ell_n/p^e}) = \frac{1}{a_\ell}$ and $\phi(x_1^{\lambda_1/p^e} \cdots x_n^{\lambda_n/p^e}) = 0$ for $(\lambda_1, \ldots, \lambda_n) \neq (\ell_1, \ldots, \ell_n)$. Then $\phi(f^{1/p^e}) = 1$, and so $\phi(J^{1/p^e}) = S$. □

We have an explicit construction of $\tau(R)$ as we did for $\tau(R,\phi)$.

Theorem 3.14. Fix any $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ and c a test element for ϕ. Then

$$\tau(R) = \sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} \psi\left((cR)^{1/p^e}\right).$$

Proof. Let J be any nonzero ideal of R for which $\psi(J^{1/p^e}) \subseteq J$ for all $e \geq 0$ and $\psi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. Let $0 \neq d \in J$, then there exists $n \in \mathbb{N}$ such that

$$c = \phi^n\left((dR)^{1/p^{ne}}\right) \subseteq \phi^n\left(J^{1/p^{ne}}\right) \subseteq J.$$

Therefore $c \in J$, and so

$$\begin{aligned}
&\sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} \psi\left((cR)^{1/p^e}\right)\\
\subseteq &\sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} \psi\left(J^{1/p^e}\right)\\
\subseteq &\sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} J\\
= &J
\end{aligned}$$

Therefore the given sum is indeed $\tau(R)$. □

Exercise 3.15. For any multiplicative system W of R, $\tau(W^{-1}R) = W^{-1}\tau(R)$.

Solution. As in exercise 3.10, for a fixed $e \geq 0$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, if c is a test element for ϕ then $\frac{c}{1}$ is a test element for $W^{-1}\phi \in \mathrm{Hom}_{W^{-1}R}((W^{-1}R)^{1/p^e}, W^{-1}R)$. Let $H = \mathrm{Hom}_{W^{-1}R}((W^{-1}R)^{1/p^e}, W^{-1}R)$. Then

$$\begin{aligned}
\tau(W^{-1}R) &= \sum_{e\geq 0} \sum_{W^{-1}\psi \in H} (W^{-1}\psi)\left(\left(\frac{c}{1}\cdot W^{-1}R\right)^{1/p^e}\right)\\
&= \sum_{e\geq 0} \sum_{W^{-1}\psi \in H} (W^{-1}\psi)\left(W^{-1}(cR)^{1/p^e}\right)\\
&= \sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} W^{-1}\psi\left((cR)^{1/p^e}\right)\\
&= W^{-1}\sum_{e\geq 0} \sum_{\psi \in \mathrm{Hom}_R(R^{1/p^e},R)} \psi\left((cR)^{1/p^e}\right)\\
&= W^{-1}\tau(R).
\end{aligned}$$ □

Exercise 3.17. Suppose that R is F-pure. Then $\tau(R)$ is radical and $R/\tau(R)$ is F-pure.

Solution. There exists a splitting $\phi : R^{1/p} \to R$ of the inclusion $R \subseteq R^{1/p}$. Suppose that $x \in R$ such that $x^{1/p} \in \tau(R)$. Then

$$x = \phi((x^p)^{1/p}) \subseteq \phi(\tau(R)^{1/p}) \subseteq \tau(R).$$

Therefore $\tau(R)$ is radical.

Define $\psi : R^{1/p} \to R/\tau(R)$ by $\psi(r^{1/p}) = \phi(r^{1/p}) + \tau(R)$. Then $r^{1/p} \in \ker\psi$ if and only if $\phi(r^{1/p}) \in \tau(R)$, and so we have that $\ker\psi \supseteq \tau(R)^{1/p}$ since $\phi(\tau(R)^{1/p}) \subseteq \tau(R)$. Therefore we can construct $\bar{\psi} : R^{1/p}/\tau(R)^{1/p} \to R/\tau(R)$ as $\bar{\psi}(r^{1/p} + \tau(R)^{1/p}) = \psi(r^{1/p})$. Now

$$\bar{\psi}(1^{1/p} + \tau(R)^{1/p}) = \psi(1^{1/p}) = \phi(1^{1/p}) + \tau(R) = 1 + \tau(R),$$

so $\bar{\psi}$ is an F-splitting of $R/\tau(R)$. □

Exercise 3.18. Suppose R is reduced and let R^N be its normalization. Let $\mathfrak{c}$ be the conductor of R in R^N, that is, $\mathfrak{c} = \mathrm{Ann}_R(R^N/R) = (R :_R R^N)$ ($\mathfrak{c}$ can also be described as the largest ideal of R^N which is also an ideal of R). Then $\tau(R) \subseteq \mathfrak{c}$.

Solution. Let $e \geq 0$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. Take $\frac{r}{s} \in R^N$ and $x \in \mathfrak{c}$. Then $\frac{r}{s}$ has an equation of integral dependence

$$\left(\frac{r}{s}\right)^n + a_1 \left(\frac{r}{s}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{r}{s}\right) + a_n = 0$$

with $a_i \in R$. Raising this to the p^eth power gives us an equation of integral dependence for $\dfrac{r^{p^e}}{s^{p^e}}$:

$$\left(\frac{r^{p^e}}{s^{p^e}}\right)^n + a_1^{p^e}\left(\frac{r^{p^e}}{s^{p^e}}\right)^{n-1} + \cdots + a_{n-1}^{p^e}\left(\frac{r^{p^e}}{s^{p^e}}\right) + a_n^{p^e} = 0.$$

Since $x \in (R :_R R^N)$, we have that $\dfrac{xr^{p^e}}{s^{p^e}} \in R$, i.e. $xr^{p^e} = x's^{p^e}$ for some $x' \in R$. Therefore,

$$\phi(x^{1/p^e})\frac{r}{s} = \phi((xr^{p^e})^{1/p^e})\frac{1}{s} = \phi((x's^{p^e})^{1/p^e})\frac{1}{s} = \phi(x'^{1/p^e})\frac{s}{s} = \phi(x'^{1/p^e}) \in R$$

Therefore $\phi(x^{1/p^e}) \in \mathfrak{c}$, hence $\phi(\mathfrak{c}^{1/p^e}) \subseteq \mathfrak{c}$, i.e. $\mathfrak{c}$ is ϕ-compatible, and so we have that $\mathfrak{c} \supseteq \tau(R)$. □

The result of the computation in Exercise 3.13 can be extended to a characterization of when $\tau(R) = R$:

Theorem 3.19. Suppose R is a domain essentially of finite type over a perfect field k. Then $\tau(R) = R$ if and only if for every $0 \neq c \in R$, there exists $e \geq 1$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ such that $\phi(c^{1/p^e}) = 1$.

The paper leaves only one step to us: to reduce to the case where R is local. Consider the R-module $R/\tau(R)$. This module is 0 (i.e. $\tau(R) = R$) if and only if $(R/\tau(R))_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ of R. However, since $(R/\tau(R))_\mathfrak{m} \cong R_\mathfrak{m}/\tau(R)_\mathfrak{m} \cong R_\mathfrak{m}/\tau(R_\mathfrak{m})$, it suffices to consider the case where R is a local ring with maximal ideal $\mathfrak{m}$.

We call a ring R such that $\tau(R) = R$ a *strongly F-regular* ring.

Theorem 3.21. A regular ring is strongly F-regular.

Exercise 3.22. A strongly F-regular ring is Cohen-Macaulay

Exercise 3.23. Suppose that $R \subseteq S$ is a split inclusion of normal domains and S is strongly F-regular. Then R is strongly F-regular and hence Cohen-Macaulay.

Solution. Let $s : S \to R$ be a splitting map for the inclusion $R \subseteq S$. Let $0 \neq c \in R$. Then $0 \neq c \in S$, and so since S is strongly F-regular, there exists $\phi \in \mathrm{Hom}_S(S^{1/p^e}, S)$ such that $\phi(c^{1/p^e}) = 1$. Since ϕ is an S-module homomorphism it is also an R-module homomorphism, so the map $\phi|_{R^{1/p^e}} \in \mathrm{Hom}_R(R^{1/p^e}, S)$ is an R-module homomorphism. Now let $\psi = s \circ \phi|_{R^{1/p^e}}$. Then $\psi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ and $\psi(c^{1/p^e}) = 1$. Therefore R is strongly F-regular. By Exercise 3.22, then, R is Cohen-Macaulay. $\square$

3.3 Test ideals in Gorenstein local rings

Exercise 3.26. Suppose that $S = k[x_1, \ldots, x_n]$, where k is a perfect field, and consider the S-linear map $\Psi : S^{1/p^e} \to S$ sending $(x_1 \cdots x_n)^{(p^e-1)/p^e}$ to 1 and all other basis elements $(x_1^{\lambda_1} \cdots x_n^{\lambda_n})^{1/p^e}$, $0 \leq \lambda_i \leq p^e - 1$ to zero. Then Ψ generates $\mathrm{Hom}_S(S^{1/p^e}, S)$ as an S^{1/p^e}-module.

Note! There is a typo in the original paper, which states that Ψ generates $\mathrm{Hom}_S(S^{1/p^e}, S)$ as an S-module, which can easily be shown false.

Solution. Let $\phi \in \mathrm{Hom}_S(S^{1/p^e}, S)$. Then for every tuple $(\lambda_1, \ldots, \lambda_n)$ with $0 \leq \lambda_i \leq p^e - 1$, there exists $a(\lambda_1, \ldots, \lambda_n) \in S$ such that ϕ takes the basis element $(x_1^{\lambda_1} \cdots x_n^{\lambda_n})^{1/p^e}$ to $a(\lambda_1, \ldots, \lambda_n)$. We claim that

$$\phi = \sum_{0 \leq \lambda_i \leq p^e-1} \left(a(\lambda_1, \ldots, \lambda_n)^{p^e} x_1^{p^e-1-\lambda_1} \cdots x_n^{p^e-1-\lambda_n} \right)^{1/p^e} \cdot \Psi.$$

Recall that the action of S^{1/p^e} on $\mathrm{Hom}_S(S^{1/p^e}, S)$ is by premultiplication; that is, for $s^{1/p^e} \in S^{1/p^e}$ and $\varphi \in \mathrm{Hom}_S(S^{1/p^e}, S)$, we have that $s^{1/p^e} \cdot \varphi \in \mathrm{Hom}_S(S^{1/p^e}, S)$ is given by $(s^{1/p^e} \cdot \varphi)(x^{1/p^e}) = \varphi(s^{1/p^e} x^{1/p^e}) = \varphi((sx)^{1/p^e})$. We now prove the above equality by checking that they act equivalently on basis elements. Let $(\lambda_1^*, \dots, \lambda_n^*)$ be a tuple with $0 \le \lambda_i \le p^e - 1$. Then

$$\left(\sum_{0 \le \lambda_i \le p^e - 1} \left(a(\lambda_1, \dots, \lambda_n)^{p^e} x_1^{p^e - 1 - \lambda_1} \cdots x_n^{p^e - 1 - \lambda_n} \right)^{1/p^e} \cdot \Psi \right) \left(\left(x_1^{\lambda_1^*} \cdots x_n^{\lambda_n^*} \right)^{1/p^e} \right)$$
$$= \sum_{0 \le \lambda_i \le p^e - 1} \Psi \left(\left(a(\lambda_1, \dots, \lambda_n)^{p^e} x_1^{p^e - 1 - \lambda_1 + \lambda_1^*} \cdots x_n^{p^e - \lambda_n + \lambda_n^*} \right)^{1/p^e} \right)$$
$$= \sum_{0 \le \lambda_i \le p^e - 1} a(\lambda_1, \dots, \lambda_n) \Psi \left(\left(x_1^{p^e - 1 - \lambda_1 + \lambda_1^*} \cdots x_n^{p^e - \lambda_n + \lambda_n^*} \right)^{1/p^e} \right)$$

By construction, the term $\Psi \left(\left(x_1^{p^e - 1 - \lambda_1 + \lambda_1^*} \cdots x_n^{p^e - \lambda_n + \lambda_n^*} \right)^{1/p^e} \right)$ is nonzero if and only if each $p^e - 1 - \lambda_i + \lambda_i^*$ is congruent to $p^e - 1$ modulo p^e. Since $0 \le \lambda_i, \lambda_i^* \le p^e - 1$, we have that $0 \le p^e - 1 - \lambda_i + \lambda_i^* \le 2p^e - 2$. Therefore this term is nonzero exactly when $p^e - 1 - \lambda_i + \lambda_i^* = p^e - 1$, i.e. when $\lambda_i = \lambda_i^*$. Therefore

$$\sum_{0 \le \lambda_i \le p^e - 1} a(\lambda_1, \dots, \lambda_n) \Psi \left(\left(x_1^{p^e - 1 - \lambda_1 + \lambda_1^*} \cdots x_n^{p^e - \lambda_n + \lambda_n^*} \right)^{1/p^e} \right)$$
$$= a(\lambda_1^*, \dots, \lambda_n^*) \Psi \left(\left(x_1^{p^e - 1} \cdots x_n^{p^e - 1} \right)^{1/p^e} \right)$$
$$= a(\lambda_1^*, \dots, \lambda_n^*)$$
$$= \phi \left(\left(x_1^{\lambda_1^*} \cdots x_n^{\lambda_n^*} \right)^{1/p^e} \right).$$

Therefore the claim is proved, and so Ψ generates $\operatorname{Hom}_S(S^{1/p^e}, S)$ as an S^{1/p^e}-module. By Lemma 3.24, Φ_S^e also generates $\operatorname{Hom}_S(S^{1/p^e}, S)$ as an S^{1/p^e}-module. Also,

$$\operatorname{Hom}_S(S^{1/p^e}, S) \cong \operatorname{Hom}_S(S^{1/p^e}, \omega_S) \cong (\omega_S)^{1/p^e} \cong S^{1/p^e},$$

and so we have that $\operatorname{Hom}_S(S^{1/p^e}, S)$ is a faithful cyclic module. Since any two generators of a faithful cyclic module differ by a unit, we have that Ψ and Φ_S^e are identical up to multplication by a unit in S^{1/p^e}. □