Research Statement

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My research is in the mathematical field of algebra, more specifically commutative algebra. Commutative algebra is the study of commutative rings, which are sets with operations of addition, subtraction, and commutative multiplication. Standard examples include rings of polynomials over a field. Much of the motivation for studying commutative algebra comes from algebraic geometry, which is the study of varieties, that is, sets defined as the points where a set of polynomials vanish. For instance, the set of points (x, y) where the polynomial $y - x^2$ vanishes is the parabola. Going in the other direction, given a collection of points, we can study the set of polynomials that vanish at every one of those points. This set is an an example of an ideal, which are subsets of rings with very nice properties, the most important one being that the quotient of an ideal by a ring is also a ring.

My long term research goals include understanding more about numerical measures of complexity and singularities of rings and ideals. In particular, I am interested in the F-threshold, which is a measure of how the regular powers of one ideal compare with the Frobenius powers of another ideal, and the F-pure threshold, which is a measure of how the ideal interacts with certain maps from the ring to itself. The F-threshold, which is defined using a limit, was recently proved to exist in all cases [6]. The notions behind the F-threshold inspired my Ph.D. advisor, Mark Johnson, and I to define an expanded notion of multiplicity connecting two related measures of the complexity of an ideal. The Hilbert-Samuel multiplicity (or just the multiplicity) of an ideal measures the behavior of an ideal when one takes quotients by larger and larger powers of it. The Hilbert-Kunz multiplicity of an ideal in a ring of positive characteristic measures the behavior of quotients by the Frobenius powers of the ideal. These two multiplicities are very interesting to me because their values can tell us information about the ring itself. For instance, if the ring is local and the multiplicity of the maximal ideal is 1, then the ring is regular, i.e. nonsingular. Studying the relationship between the normal powers and the Frobenius powers of ideals allowed me to create a function that interpolates between the two values and gives a common framework in which to express statements about the multiplicities.

Current Research

Most of my research has been on rings of positive characteristic, that is, rings R in which $p \cdot 1 = 0$ for some positive prime number p, where 1 is the multiplicative identity of R. Because of this structure, it is the case that if x and y are any two elements of R, then

$$(x + y)^p = x^p + y^p$$

a behavior not found in rings of characteristic 0. This fact means that the Frobenius function F taking $x \in R$ to x^p is a ring homomorphism, which in turn allows us to define a new R-module structure on R. A famous theorem of Kunz [2] says that the ring R is regular if and only if R is a free R-module under this new structure. As regular rings are precisely those corresponding to smooth varieties, there is a great deal of interest in understanding the nature of the function F and its interaction with the ring R.

F-Singularities of Generic Links. In June 2015, I attended the AMS Mathematics Research Communities workshop in Snowbird, Utah, where I worked with Linquan Ma, Janet Page, Rebecca R.G., and Wenliang Zhang to take a characteristic 0 theorem of Niu [4] and prove it in characteristic p. The theorem we proved had to do with studying the F-singularities of the generic link of an ideal.

We say that two ideals I and J of a ring R are *linked* if there exists a regular sequence $(a_1, \dots, a_c) \subseteq I$ such that c is the height of I and $J = ((a_1, \dots, a_c) : I)$. Many properties of I imply similar or related properties for J when the ideals are linked. Finding a regular sequence $a_1, \dots, a_c \subseteq I$ can be difficult so as an alternate approach we can construct such a sequence by passing to a larger ring $S = R[u_{i,j}]$. Here we adjoin an appropriate number of variables in order to take the a_i to be generic combinations of the generators of I; that is, each a_i is a dot product of a row of the matrix $(u_{i,j})$ with the generators of I. Now we can take $J = ((a_1, \dots, a_c) : IS) \subseteq IS$, and J is indeed linked to IS. We call J a *generic link* of I.

In our paper [3], we prove that if an ideal I with height c has a reduction that is an (almost) complete intersection and $\tau(I^c) = I$ and J is any generic link of I, then S/J is F-rational. Here $\tau(I^c)$ is the (big) test ideal of the pair (R, I^c). Also in [3], we found a relationship between the F-pure thresholds of I and J, again inspired by a result of Niu on the log-canonical threshold of generic links.

Generic Residual Intersections. My work at the MRC motivated my thesis research in two different but related directions. Many of the ideas that we had used in writing the paper could be generalized to the case of generic residual intersection, which is a generalization of generic linkage. The ideal J is said to be an s-residual intersection of I if there exists an ideal $(a_1, \dots, a_s) \subseteq I$ such that $J = ((a_1, \dots, a_s) : I)$ and $\mathrm{ht} J \geq s \geq \mathrm{ht} I$. We can attempt to construct generic s-residual intersections in the same way that we construct generic links, but the condition on the height of the ideal J means that not all such constructions succeed. In [1], Huneke and Ulrich outlined a set of conditions on the ideal I that would guarantee that generic s-residual intersections could be constructed; in fact, they gave conditions for the generic residual intersection to be a *geometric* residual intersection, which is a stronger condition. In my thesis I found a weaker set of conditions on I that still guarantee that it has a generic s-residual intersection, and then prove an inequality similar to the one in [3] comparing the F-pure threshold of J to that of I.

In working on this problem I realized that the arguments that I was using to compute the F-pure threshold were actually computing the F-threshold. These two quantities are the same for ideals in regular rings, which was the setting that we had been working in. With this realization, I found that my results, if stated in terms of the F-threshold, held in a much more general context. It was at this point that I started becoming seriously interested in the F-threshold as a numerical measure of complexity.

s-multiplicity. Suppose $(R, \mathfrak{m})$ is a ring of positive characteristic p. The proof of the existence of the F-threshold inspired me to define a function $e_s(I, J)$ that would interpolate between the Hilbert-Samuel multiplicity of I and the Hilbert-Kunz multiplicity of J as a real parameter s varies as follows:

$$e_s(I, J) = \lim_{e \to \infty} \frac{\lambda\left(R/(I^{\lceil sp^e \rceil} + J^{[p^e]})\right)}{q^d \mathcal{E}_d(s)}$$

where $\mathcal{E}_d(s)$ is a normalizing function with a recursive formula such that if R is a regular ring, $e_s(\mathfrak{m}, \mathfrak{m}) = 1$ for all s. In my thesis I proved that this limit always exists for $\mathfrak{m}$-primary ideals I and J.

The interpolation between Hilbert-Samuel multiplicity and Hilbert-Kunz multiplicity has some nice properties. In particular, $e_s(I, J) = e_{\mathrm{HK}}(J)$, the Hilbert-Kunz multiplicity of J, whenever s is greater than both the dimension of the ring and the F-threshold of I with respect to J, and $e_s(I, J) = e(I)$, the Hilbert-Samuel multiplicity of I, whenever s is less than both 1 and a threshold based on a dual notion to that of the F-threshold. As a function of s, $e_s(I, J)$ is continuous and bounded above by $\lambda(R/\mathfrak{n})$, where $\mathfrak{n}$ is a minimal reduction of R, assuming R has infinite residue field.

As part of my thesis I was able to provide a method for computing the s-multiplicity of a pair of monomial ideals in toric space by expressing it a volume in d-dimensional real space space. This construction, being very visual, helped to build intuition and understanding of the nature of the s-multiplicity function. As a corollary to this construction, I gave a new proof that the Hilbert-Kunz multiplicity of a toric ring is rational.

Hilbert-Samuel multiplicity is closely related to integral closure of ideals, and Hilbert-Kunz multiplicity is closely related to tight closure of ideals, and so it was natural to try to find a family of closure operators that would be related in the same way to s-multiplicity. The most straightforward definition of such an operator cl_s (for a fixed $s \geq 1$) was the following:

$$x \in I^{\mathrm{cl}_s} \Leftrightarrow \exists c \in R^\circ \text{ such that } \forall e \gg 0, cx^{p^e} \in I^{\lceil sp^e \rceil} + I^{[p^e]}$$

with this definition, for $s = 1$, the operation cl_s is integral closure, and for large s the operation is tight closure. I proved that if $x \in I^{\mathrm{cl}_s}$, then $e_s((I, x), (I, x)) = e_s(I, I)$, recovering the analogous statement for Hilbert-Samuel (resp. Hilbert-Kunz) multiplicity and integral (resp. tight) closure.

More interesting is the converse direction: if $J \subseteq I$ and $e_s(J, J) = e_s(I, I)$, does $J^{\mathrm{cl}_s} = I^{\mathrm{cl}_s}$? I was able to prove this via a theoerem of Polstra and Tucker in [5] in the case that the ring R is an F-finite complete domain. Extending this to the complete (not necessarily domain) case required two steps. The first was the development of a general Associativity Formula for s-multiplicity:

$$e_s(I, J) = \sum_{\mathfrak{p} \in \mathrm{Assh}\, R} e_s(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}),$$

where Assh R denotes the set of prime ideals $\mathfrak{p} \subseteq R$ such that $\dim R/\mathfrak{p} = \dim R$. The second step was to establish that $x \in I^{\mathrm{cl}_s}$ if and only if $x + \mathfrak{p} \in (I + \mathfrak{p})^{\mathrm{cl}_s}$ for every $\mathfrak{p}$ a minimal prime of R. The conclusion, then, is that for an F-finite complete equidimensional ring, $x \in I^{\mathrm{cl}_s}$ if and only if $e_s((I, x), (I, x)) = e_s(I, I)$.

Future Research

Moving forward I have a few main research tracks I wish to pursue. The first is to understand more about the s-multiplicity and what it tells us about ideals and rings of positive characteristic. In particular, I wish to know the answer to the following questions:

Question 1 If $e_s(\mathfrak{m}, \mathfrak{m}) = 1$ for some $s \geq 0$, is the ring regular? Under what conditions do we have an affirmative answer?

This question is based on the fact that for a reasonably nice ring (equidimensional is sufficient), if either the Hilbert-Samuel or the Hilbert-Kunz multiplicities are equal to 1, then the ring is regular. I would like to know if it suffices for the s-multiplicity to be 1 for any s.

Question 2 Is there a natural way to extend the notion of s-multiplicity to non-$\mathfrak{m}$-primary ideals?

There have been several explorations of how to define Hilbert-Samuel and Hilbert-Kunz multiplicities for non-$\mathfrak{m}$-primary ideals. If I could find a reasonable, natural way to extend s-multiplicity to those cases as well, it could provide a unifying context to work in. A psooible first step is to examine what happens when only one of I and J are $\mathfrak{m}$-primary. The existence of the limit defining $e_s(I,J)$ can still be shown to exist using a theorem of Polstra and Tucker in [5], so this may provide some insight.

Question 3 What can be said about the s-function, that is, the function H_s that sets $H_s(p^e) = \lambda\left(R/(I^{\lceil sp^e \rceil} + J^{[p^e]})\right)$?

Based on the work already done, we know that $H_s(p^e) \in \mathcal{O}(p^{ed})$, where $d = \dim R$, but beyond that little is yet known. We have that H_s will be (essentially) of polynomial type for small values of s, since there it equals the Hilbert function of I. However, for large values of s it will equal the Hilbert-Kunz function of J, which may not be of polynomial type. Thus, we can ask: for what values of s is H_s polynomial-type? For a fixed e, is $H_s(p^e)$ a continuous function of s?

Question 4 How can we relate algebraic and analytic notions like multiplicity to geometric notions, especially volume?

Recently, several researchers have used a volume construction to compute various numerical measures of complexity, for instance, Daniel Hernández and Jack Jeffries have used volumes to compute the Hilbert-Kunz multiplicity and F-signature for monomial ideals. In my thesis I have used volumes to compute s-multiplicities for binomial hypersurfaces. Is there a general framework that allows questions to be framed in terms of easily-computed geometry?

Numerical measures of complexity are a common theme throughout my research because I appreciate the detailed information that they give about the rings and ideals involved. The fact that a multiplicity can detect even a small change in a set of generators makes it a useful tool for comparing and contrasting the algebraic objects that we work with. I look forward to continuing to study how to carefully categorize and measure rings and ideals as I move forward in my career.

References

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