Research Statement

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My research is in the mathematical field of algebra, specifically commutative algebra. Much of the motivation for studying commutative algebra comes from algebraic geometry, which is the study of varieties, that is, sets defined as the points where a set of polynomials vanish. For instance, the set of points (x, y) where the polynomial $y - x^2$ vanishes is the parabola. Going in the other direction, given a collection of points, we can study the set of polynomials that vanish at every one of those points. This set is an an example of an ideal, which are subsets of rings with very nice properties, the most important one being that the quotient of an ideal by a ring is also a ring. My approach to these problems is primarily an algebraic one, though I am always looking for ways to apply my work to algebraic geometry or vice versa.

A specific area that I have worked in is numerical invariants of rings and ideals. Various numerical measures have been defined which correspond to certain nice properties of an algebraic object. For a recent example, the F-signature of a ring of positive characteristic is a value between 0 and 1 having the following properties under very mild conditions: the ring is regular, i.e. describes a smooth set of points in space, if and only if the F-signature is 1; and the ring is strongly F-regular, a condition which is the positive characteristic analogue of the log terminal condition in characteristic 0, if and only if the F-signature is positive. Other examples of such numerical measures are the F-pure threshold, the F-threshold, and the Hilbert-Samuel and Hilbert-Kunz multiplicities. These last two invariants were the main focus of my dissertation work, and understanding their relationship required understanding more about the F-threshold and F-pure threshold.

Multiplicity and Lech's Conjecture

For a local ring $(R, \mathfrak{m})$ and an ideal $I \subseteq R$, the Hilbert-Samuel multiplicity of I is defined to be $e(I) = \lim_{n\to\infty} d! \cdot \lambda(R/I^n)/n^d$, where $d = \dim R$ and $\lambda(M)$ is the length of an R-module M. Of particular interest is $e(\mathfrak{m})$, the Hilbert-Samuel multiplicity of the maximal ideal itself, which we also denote by $e(R)$. This value, which is always a positive integer, carries information about the structure of the ring. For instance, if R is a regular ring, then $e(R) = 1$. A classical theorem of Nagata proves that if R is equidimensional, the converse holds as well.

One of the most elusive open questions involving Hilbert-Samuel multiplicity is known as Lech's Conjecture. More than 50 years ago, in [3], Lech considered the situation where $R \subseteq S$ is a flat extension of local rings. Lech proved that if $\dim R \leq 2$, then $e(R) \leq e(S)$, and conjectured that the same inequality would hold in all dimensions. Since that time, there has been only one improvement along those lines: a 2016 paper, [4], by Linquan Ma, in which Lech's Conjecture is proven in dimension 3 if the rings are of equal characteristic. Ma's paper reduces the problem to the positive characteristic case, where certain techniques make the problem more tractable. In particular, Ma related the quantities involved to an analogue of the Hilbert-Samuel multiplicity known as the Hilbert-Kunz multiplicity.

If a ring R is of positive prime characteristic p, and $I \subseteq R$ is an ideal, then the ideal generated by elements of the form i^{p^e} for $i \in I$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of I. The Frobenius power $I^{[p^e]}$ is contained in I^{p^e} and is usually much smaller. The Hilbert-Kunz multiplicity of I is $e_{HK}(I) = \lim_{e\to\infty} \lambda(R/I^{[p^e]})/p^{ed}$. Like the Hilbert-Samuel multiplicity, the Hilbert-Kunz multiplicity is a real number at least 1, though unlike the Hilbert-Samuel multiplicity it can be non-integer, and even irrational. The Hilbert-Kunz multiplicity is a more sensitive measure of singularity than the Hilbert-Samuel multiplicity. Many theorems true of Hilbert-Samuel multiplicity have analogous versions for Hilbert-Kunz multiplicity, though the Hilbert-Kunz versions are often more difficult to prove. A notable exception to this pattern is Lech's Conjecture. In fact, it is known that $e_{HK}(R) \leq e_{HK}(S)$ whenever $R \subseteq S$ is a flat inclusion of local rings. Therefore, one possible way of attacking Lech's Conjecture is to understand more fully the relationship between the two multiplicities.

Interpolating Between Multiplicities

To relate Hilbert-Samuel and Hilbert-Kunz multiplicities, I investigated a function that interpolates between the Hilbert-Samuel multiplicity of I and the Hilbert-Kunz multiplicity of J as a positive real parameter s varies, for two ideals I and J of a ring R. The function is defined by

$$e_s(I,J) = \lim_{e\to\infty} \frac{\lambda\left(R/(I^{\lceil sp^e\rceil} + J^{[p^e]})\right)}{p^{ed}\mathcal{H}_s(d)}$$

where $\mathcal{H}_s(d)$ is a normalizing function which guarantees that if R is a regular ring, then $e_s(\mathfrak{m},\mathfrak{m}) = 1$ for all s. In [13] I proved that this limit always exists for $\mathfrak{m}$-primary ideals I and J, and established that for each s, this function satisfies many of the properties that Hilbert-Samuel and Hilbert-Kunz multiplicities. This function does indeed interpolate between the two multiplicities in a way that is related two two other limits in positive characteristic. In particular, $e_s(I,J) = e_{\mathrm{HK}}(J)$ whenever s is greater than both the dimension of the ring and the F-threshold of I with respect to J, a value defined in [9] and proved to exist in [1]. On the other end of the spectrum, $e_s(I,J) = e(I)$ whenever s is less than both 1 and a threshold based on a dual notion to that of the F-threshold. As a function of s, $e_s(I,J)$ is continuous.

As part of my work I was able to provide a method for computing the s-multiplicity of a pair of monomial ideals in toric space by expressing it as a volume in d-dimensional real space. This construction, being very visual, helped to build intuition and understanding of the nature of the s-multiplicity function. A corollary to this construction is a new proof that the Hilbert-Kunz multiplicity of a toric ring is rational.

I collaborated with Lance E. Miller to write two papers related to s-multiplicity. In [7], we built upon several arguments of Watanabe and Yoshida to establish bounds for s-multiplicity in terms of the Hilbert-Samuel and Hilbert-Kunz multiplicities. For instance, we showed that $e_s(I,I)$ is constant in s if and only if $e(I) = e_{HK}(I)$. If R is Cohen-Macaulay, then we recover a significantly generalized version of a lower bound established in [14]. In particular, if J is a parameter ideal reduction of I, then for any $1 \leq t \leq s$, we have that

$$e_s(I) \geq \left(\frac{\mathcal{H}_t(d) - \mu(I/J^*)\mathcal{H}_{t-1}(d)}{\mathcal{H}_s(d)}\right) e(I).$$

We also examined an s-multiplicity version of a famous conjecture of Watanabe and Yoshida concerning the minimal values of $e_{HK}(R)$ for singular rings. In particular, we considered the following question: If R is a singular ring of dimension d, is it the case that $e_s(R) \geq e_s(R_d)$, where $R_d = k[[x_0,\ldots,x_d]]/(\sum_i x_i^2)$? We were able to establish a positive answer to this question in the Cohen-Macaulay case in dimension 3 or less and in the complete intersection case when $p \geq 3$.

In [8], we examined the values $\ell(p^e) = \lambda(R/(\mathfrak{m}^{\lceil sp^e \rceil} + \mathfrak{m}^{[p^e]}))$ for $R = k[X]/I_2(X)$, where X is a matrix of indeterminates and $I_2(X)$ is the ideal generated by all 2×2 minors of X. We were expanding upon the results appearing in [6] and [12] on the Hilbert-Kunz function. Using Gröbner basis arguments and combinatorial arguments, we derived a closed form for $\ell(p^e)$. Two consequences of this result are a way of calculating the s-multiplicity of 2×2 determinantal rings by examining the leading term of this closed form, and a proof that if $s \in \mathbb{Z}[p^{-1}]$, then $\ell(p^e)$ is eventually polynomial in p^e. This second result is notable since in general the Hilbert-Samuel function $\lambda(R/\mathfrak{m}^n)$ is a polynomial in n but the Hilbert-Kunz function $\lambda(R/\mathfrak{m}^{[p^e]})$ is not a polynomial in p^e.

s-Closures

Hilbert-Samuel multiplicity is closely related to integral closure of ideals, and Hilbert-Kunz multiplicity is closely related to tight closure of ideals, so it was natural to try to find a family of closure operators that would be related in the same way to s-multiplicity. The most straightforward definition of such an operator cl_s (for a fixed $s \geq 1$) was to say that $x \in I^{\mathrm{cl}_s}$ if there exists some c not in any minimal prime of R such that for all $e \gg 0$, we have that $cx^{p^e} \in I^{\lceil sp^e \rceil} + I^{[p^e]}$. For $s = 1$, the operation cl_s is integral closure, and for large s the operation is tight closure. I proved that if $x \in I^{\mathrm{cl}_s}$, then $e_s((I,x),(I,x)) = e_s(I,I)$, recovering the analogous statement for Hilbert-Samuel (resp. Hilbert-Kunz) multiplicity and integral (resp. tight) closure.

Also interesting is the converse direction: if $J \subseteq I$ and $e_s(J,J) = e_s(I,I)$, does $J^{\mathrm{cl}_s} = I^{\mathrm{cl}_s}$? I was able to prove this via a theorem of Polstra and Tucker in [11] in the case that the ring R is an F-finite complete domain. Extending this to the complete equidimensional (not necessarily domain) case requires two ingredients. The first is the development of a general Associativity Formula for s-multiplicity:

$$e_s(I,J) = \sum_{\mathfrak{p} \in \mathrm{Assh}\, R} e_s(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}),$$

where $\mathrm{Assh}\, R$ denotes the set of prime ideals $\mathfrak{p} \subseteq R$ such that $\dim R/\mathfrak{p} = \dim R$. I proved this statement in [13], using the fact that $e_s(I,J)$ is continuous in s to establish that s-multiplicity is additive on short exact sequences

The second required ingredient is knowing that we can test for s-closure modulo minimal primes. That is, we need that $x \in I^{\mathrm{cl}_s}$ if and only if $x + \mathfrak{p} \in (I + \mathfrak{p})^{\mathrm{cl}_s}$ for every minimal prime $\mathfrak{p}$ of R. The corresponding statement for integral closure can be proved using an argument involving an equation of integral dependence, which we do not have in this case. The analogous statement for tight closure is proven using a theory of test elements, which we do not have for the larger s-closures. Thus proving this second fact will require a novel argument, and is one of my primary goals in the direction of s-closures.

F-Singularities of Generic Residual Intersections

In June 2015, I attended the AMS Mathematics Research Communities workshop in Snowbird, Utah, where I worked with Linquan Ma, Janet Page, Rebecca R.G., and Wenliang Zhang to take a characteristic 0 theorem of Niu [10] and prove the analogous statement in positive characteristic. The theorem we proved had to do with studying the F-singularities of the generic link of an ideal.

We say that two ideals I and J of a ring R are *linked* if there exists a regular sequence $(a_1, \ldots, a_c) \subseteq I$ such that c is the height of I and $J = ((a_1, \ldots, a_c) : I)$. Many properties of I imply similar or related properties for J when the ideals are linked. Finding a regular sequence $a_1, \ldots, a_c \subseteq I$ can be difficult, so as an alternate approach we can construct such a sequence by passing to a larger ring $S = R[u_{i,j}]$. Here we adjoin an appropriate number of variables in order to take the a_i to be generic combinations of the generators of I; that is, each a_i is a dot product of a row of the matrix $(u_{i,j})$ with the generators of I. Now we can take $J = ((a_1, \ldots, a_c) : IS) \subseteq S$, and J is indeed linked to IS. We call J a *generic link* of I.

In our paper [5], we prove that if an ideal I with height c has a reduction that is an (almost) complete intersection, $\tau(I^c) = I$, and J is any generic link of I, then S/J is F-rational. Here $\tau(I^c)$ is the (big) test ideal of the pair (R, I^c). Also in [5], we found a relationship between the F-pure thresholds of I and J, again inspired by a result of Niu on the log-canonical threshold of generic links.

My work at the MRC motivated my research in two different but related directions. Many of the ideas that we had used in writing the paper could be generalized to the case of generic residual intersection, which is a generalization of generic linkage. The ideal J is said to be an s-residual intersection of I if there exists an ideal $(a_1, \ldots, a_s) \subseteq I$ such that $J = ((a_1, \ldots, a_s) : I)$ and $\mathrm{ht}J \geq s \geq \mathrm{ht}I$. We can attempt to construct generic s-residual intersections in the same way that we construct generic links, but the condition on the height of the ideal J means that not all such constructions succeed. In [2], Huneke and Ulrich outlined a set of conditions on the ideal I that would guarantee that generic s-residual intersections could be constructed; in fact, they gave conditions for the generic residual intersection to be a *geometric* residual intersection, which is a stronger condition. During my graduate work I found a weaker set of conditions on I that still guarantee that it has a generic s-residual intersection, and then proved an inequality similar to the one in [5] comparing the F-pure threshold of J to that of I.

In working on this problem I realized that the arguments that I was using to compute the F-pure threshold were actually computing the F-threshold. These two quantities are the same for ideals in regular rings, which was the setting that we had been working in. With this realization, I found that my results, if stated in terms of the F-threshold, held in a much more general context. It was at this point that I started becoming seriously interested in the F-threshold as a numerical measure of complexity.

Future Research

Moving forward I have a few main research tracks I wish to pursue. The first is to understand more about the s-multiplicity and what it tells us about ideals and rings of positive characteristic. In particular, I wish to know the answer to the following questions:

Question 1: Can we use the fact that Lech's Conjecture holds for Hilbert-Kunz multiplicity and the continuously interpolating nature of $e_s(I, J)$ to conclude that Lech's Conjecture holds in more cases? As a first step in that direction, can we reduce the conjecture to a more tractable problem based on the interpolation? Lech's Conjecture is one of the longest standing open problems in commutative algebra and even partial results are interesting.

Question 2: If $e_s(\mathfrak{m}, \mathfrak{m}) = 1$ for some $s \geq 0$, is the ring regular? This question is based on the fact that for a reasonably nice ring (equidimensional is sufficient), if either the Hilbert-Samuel or the Hilbert-Kunz multiplicities are equal to 1, then the ring is regular. I would like to know if it suffices for the s-multiplicity to be 1 for any s.

Question 3: Is there a natural way to extend the notion of s-multiplicity to non-$\mathfrak{m}$-primary ideals? There have been several explorations of how to define Hilbert-Samuel and Hilbert-Kunz multiplicities for non-$\mathfrak{m}$-primary ideals. If I could find a reasonable, natural way to extend s-multiplicity to those cases as well, it could provide a unifying context to work in. A possible first step is to examine what happens when only one of I and J are $\mathfrak{m}$-primary. The existence of the limit defining $e_s(I, J)$ can still be shown in broad cases using a theorem of Polstra and Tucker in [11], so this may provide some insight.

Question 4: How do the natural interpolating analogues of other numerical measures for rings of positive characteristic behave? For example, using techniques in [11], we can define an interpolating form of the F-signature, a value closely related to Hilbert-Kunz multiplicity. This is a relatively new question in my research, and one that I am very interested in pursuing.

Question 5: How can we relate algebraic and analytic notions like multiplicity to geometric notions, especially volume? Recently, several people have used a volume construction to compute various numerical measures of complexity. For instance, Daniel Hernández and Jack Jeffries have used volumes to compute the Hilbert-Kunz multiplicity and F-signature for monomial ideals. In my thesis I have used volumes to compute s-multiplicities for binomial hypersurfaces. Is there a general framework that allows questions to be framed in terms of easily-computed geometry?

Numerical measures of complexity are a common theme throughout my research because I appreciate the detailed information that they give about the rings and ideals involved. The fact that a multiplicity can detect even a small change in an ideal makes it a useful tool for comparing and contrasting the algebraic objects that we work with. I look forward to continuing to study how to carefully categorize and measure rings and ideals as I move forward in my career.

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